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REFLECTION AND TRANSMISSION COEFFICIENTS FOR TRANSVERSELY ISOTROPIC MEDIA

BY

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ABSTRACT

It has become evident in seismology and seismic prospecting that to obtain more consistent results with actual field data a more sophisticated mathematical model than that of an isotropic homogeneous earth must be considered. To this end asymptotic ray theory applied to anisotropic inhomogeneous media, has been presented by Cerveny (1), with the special case of transverse isotropy dealt with in the second of his quoted papers (2). With the object in mind of using these results to produce synthetic seismograms for transversely isotropic plane layered media it becomes necessary to compute first, reflection and transmission coefficients of the coupled quasi-compressional (P) and quasi-shear (SV) waves at an interface of two such media. These coefficients, along with the free interface case will be presented here.

As the requirements for continuity of displacements and stresses will be developed in terms of asymptotic ray theory, a brief treatment of it (after Hron and Kanasewich (6)) will be presented first. Elements of the theory of Cerveny's papers (1),(2) will be quoted, in so far as it is necessary to obtain expressions for the normal velocities of the wavefronts.

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ASYMPTOTIC RAY THEORY FOR AN ANISOTROPIC INHOMOGENEOUS MEDIA

In general the equations of motion in an elastic media, in Cartesian coordinates are (neglecting body forces)

$$(1.1) \qquad \rho \ddot{u}_i = \frac{\partial \sigma_{ij}}{\partial x_j}$$

where $\partial\sigma_{ij}/\partial x_j$ is the internal stress force, ρ is density and $\ddot{u}_i$ is acceleration.

The stress tensor σ_{ij} is related to the elastic modulus tensor $c_{ijk\ell}$ (which is in general coordinate dependent) as

$$(1.2) \qquad \sigma_{ij} = c_{ijk\ell} u_{k\ell}$$

where

$$u_{k\ell} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_\ell}{\partial x_k}\right) \quad .$$

The elastic parameters $c_{ijk\ell}$ satisfy the conditions of symmetry

$$c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij}$$

reducing the number of independent elements of the elastic modulus tensor from 81 to 21. Substituting (1.2)

into (1.1) and using the symmetry properties of the $c_{ijk\ell}$ yields

$$\rho\ddot{u}_j = \frac{\partial}{\partial x_i}\left(c_{ijk\ell}\frac{\partial u_k}{\partial x_\ell}\right) \quad . \tag{1.3}$$

It will be assumed that the solution of (1.3) near the wavefront can be written generally as

$$u_i(x_m,t) = W_i(x_m,t)s(t-\tau(x_m)) + \ldots \qquad m=1,2,3 \tag{1.4}$$

where s describes the source as a function of time and may be discontinuous at $t = 0$. W_i is assumed analytic in the neighbourhood of the wavefront. The equation describing the position of the wavefront at time t is

$$t = \tau(x_m) \qquad m = 1,2,3$$

τ being the phase function which depends upon the ray path from the source to the observer at $M(x_m)$.

Let $\Delta\tau = t - \tau$ be the time a short distance from the wavefront and expand W_i in a Taylor series about $t = \tau + \Delta\tau$. This Taylor series expansion is justified only if there are no fluctuations of great magnitude in the elastic parameters of the medium in a distance of one wavelength. As it will be shown later that the normal velocity to the wavefront is dependent on the elastic parameters, the expansion can be assumed valid if the gradient of the normal velocity is slowly varying. (1.4) becomes

$$(1.5)\qquad u_i(x_m,t) = \sum_{n=0}^{\infty} \frac{1}{n!} \left\{\frac{\partial^n W_i}{\partial t^n}\right\}_{t=\tau(x_m)} (\Delta\tau)^n [s(t-\tau(x_m))]$$

{ } is totally spacially dependent while [ ] is time frequency and spacially dependent.

A monochromatic sinusoidal solution of (1.3) is

$$(1.6)\qquad u_i(x_m,t) = a_i(x_m,\omega)e^{i\omega(t-\tau)} \; .$$

Assuming a_i can be expanded in a power series in $i\omega$ yields

$$a_i = \sum_{n=0}^{\infty} \frac{A_i^{(n)}}{(i\omega)^n}$$

and (1.6) becomes

$$(1.7)\qquad u_i(x_m,t) = \sum_{n=0}^{\infty} A_i^{(n)}(x_m) \frac{e^{i\omega(t-\tau)}}{(i\omega)^n} \; .$$

A particular case for harmonic waves is when s in equation (1.5) is

$$\frac{(\Delta\tau)^n}{n!} s(t-\tau) = \frac{e^{i\omega(t-\tau)}}{(i\omega)^n} = s_n(t-\tau) \; .$$

Equation (1.6) is in a useful form since the $A_i^{(n)}$ are independent of frequency with the frequency dependent position of both the source function and wave modification due to the medium contained in the term $s_n(t-\tau(x_m))$. Equation (1.7) can thus be written

$$(1.8)\qquad u_i(x_m,t) = \sum_{n=0}^{\infty} A_i^{(n)}(x_m)s_n(t-\tau(x_m)) \; .$$

Equation (1.8) can be generalized to an impulsive source function by first obtaining the Fourier transform of the desired impulse response s(t)

i.e. $$\tilde{S}(\omega) = \int_{-\infty}^{\infty} s(t)e^{-i\omega t}\,dt\ .$$

After summing (1.8) over all frequencies, the displacement at any location is given by

$$u_i(x_m,t) = \frac{\mathrm{Re}}{\pi} \sum_{n=0}^{\infty} A_i^{(n)}(x_m) \int_{\omega_o}^{\infty} \frac{\tilde{S}(\omega)e^{i\omega(t-\tau)}}{(i\omega)^n}$$

as $\tilde{S}(\omega)$ is required to be negligible for $\omega < \omega_o$ (see high frequency approximation when the low frequency part of the source spectrum is assumed to be small compared with the rest of the spectrum). It should be noted that

$$\frac{ds_n(\xi)}{d\xi} = s_{n-1}(\xi)\ .$$

Substituting (1.8) into (1.3) yields upon manipulation of the summations, along with the additional assumption that $A_i^{(-2)} = A_i^{(-1)} \equiv 0$, $i = 1,2,3$, that

(1.9) $$s_n(t-\tau)\Big(\{A_j^{(n)} - a_{ijk\ell}p_i p_\ell A_k^{(n)}\} + \{a_{ijk\ell}p_i \frac{\partial A_k^{(n-1)}}{\partial x_\ell} + \frac{1}{\rho}\frac{\partial}{\partial x_i}(\rho a_{ijk\ell}p_\ell A_k^{(n-1)})\} - \{\frac{1}{\rho}\frac{\partial}{\partial x_i}(\rho a_{ijk\ell}\frac{\partial A_k^{(n-2)}}{\partial x_\ell})\}\Big) = 0$$

where $a_{ijk\ell} = c_{ijk\ell}/\rho$ and $p_i = \partial\tau/\partial x_i$ the slowness vector.

The p_i's at a given point and time are related to the normal velocity of wavefront at a given point and time by the relation

$$p_i = \frac{N_i}{V} ,$$

N_i being the directional cosines of the wavefront normal and V the normal velocity. For (1.9) to be satisfied, the square bracketed term must be equal to zero for all values of n, i.e.

$$(1.10) \quad \vec{N}(\vec{A}^{(n)}) - \vec{M}(\vec{A}^{(n-1)}) + \vec{L}(\vec{A}^{(n-2)}) = 0$$

$$N_j(\vec{A}^{(n)}) = \Gamma_{jk}A_k^{(n)} - A_j^{(n)}$$

$$M_j(\vec{A}^{(n-1)}) = a_{ijk\ell}p_i \frac{\partial A_k^{(n-1)}}{\partial x_\ell} + \frac{1}{\rho} \frac{\partial}{\partial x_i}\{\rho a_{ijk\ell}p_\ell A_k^{(n-1)}\}$$

$$L_j(\vec{A}^{(n-2)}) = \frac{1}{\rho} \frac{\partial}{\partial x_i} \{\rho a_{ijk\ell} \frac{\partial A_k^{(n-2)}}{\partial x_\ell}\}$$

$$\Gamma_{jk} = a_{ijk\ell}p_i p_\ell \quad .$$

For n = 0 using the fact that $\vec{A}^{(-2)} = \vec{A}^{(-1)} \equiv 0$, (1.10) becomes

$$\Gamma_{jk}A_k^{(o)} - A_j^{(o)} = 0$$

or

$$(\Gamma_{jk} - \delta_{jk})A_k^{(o)} = 0 \quad .$$

Assuming that $A_k^{(o)}$ is not equal to zero, k = 1,2,3, it follows that

(1.11) $\det(\Gamma_{jk} - \delta_{jk}) = 0$.

By definition the tensor $a_{ijk\ell}$ has the symmetry property

$$a_{ijk\ell} = a_{k\ell ij} = a_{\ell kji} \ .$$

Therefore Γ_{jk} is symmetric and as a result the eigenvalues of Γ_{jk} are all real. Since the strain energy is positive definite, Γ_{jk} is positive definite and thus the eigenvalues of Γ_{jk} are all positive.

From this it can be seen that

(1.12) $\det(\Gamma_{jk} - G\delta_{jk}) = 0$

yields 3 real and positive eigenvalues G_1, G_2, G_3. One eigenvalue corresponds to a quasi-compressional (P) wave and the other two correspond to two independent quasi-shear waves, SV and SH.

The only non-trivial solution of $(\Gamma_{jk} - \delta_{jk})A_k^{(o)} = 0$ occurs when one of the eigenvalues G_s, $s = 1,2,3$ of Γ_{jk} is equal to unity. Neglecting degenerate cases, there are three possibilities, corresponding to the propagation of one of the three different types of wavefronts. It is from these eigenvalues using the relation $p_i = N_i/V_s$, $s = 1,2,3$, that the normal velocities to the wavefronts are found.

CHAPTER 2

THE SPECIAL CASE OF THE TRANSVERSELY ISOTROPIC MEDIUM

Because of the symmetries of the $a_{ijk\ell}$, a more convenient notation arrangement may be adopted. As a result $a_{ijk\ell} \rightarrow A_{mn}$ according to the following scheme:

$$\begin{array}{ccccccccc} 11 & 22 & 33 & 23 & 32 & 13 & 31 & 12 & 21 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 4 & 5 & 5 & 6 & 6 \end{array}$$

The elastic parameters of a medium can then be tabulated in a 6×6 symmetric matrix with 21 independent entries. However, any calculations with all of these 21 parameters becomes unwieldy so some simplifying assumptions must be made. Cerveny (2) suggests a model of an isotropic inhomogeneous medium which can be described by the 9 independent parameters.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{13} & A_{23} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A66 \end{pmatrix}$$

A special case of the model, transverse isotropy, will be considered here, for which $A_{11} = A_{22}$, $A_{44} = A_{55}$, $A_{23} = A_{13}$, $A_{12} = A_{11} - 2A_{66}$. Inherent to this special case are the additional assumptions:

(a) The elastic parameters depend on only two coordinates which specify epicentral distance (x), and depth (z).

(b) The slowness vector has only two components, p_2 chosen to be identically zero.

Under these assumptions, the source lies in the plane (x,z) as does the ray under consideration for all time as well as the corresponding normal to the wavefront.

For transverse isotropy the matrix elements Γ_{jk} are given by

$$\Gamma_{11} = p_1^2 A_{11} + p_3^2 A_{55}$$

$$\Gamma_{22} = p_1^2 A_{66} + p_3^2 A_{55}$$

$$\Gamma_{33} = p_1^2 A_{55} + p_3^2 A_{33}$$

$$\Gamma_{12} = \Gamma_{21} = \Gamma_{23} = \Gamma_{32} = 0$$

$$\Gamma_{13} = \Gamma_{31} = p_1 p_3 (A_{13} + A_{55})$$

and the resultant eigenvalue problem (1.11) $\Gamma_{jk} - G\delta_{jk} = 0$ becomes

$$(2.1)\qquad \begin{pmatrix} p_1^2A_{11}+p_3^2A_{55}-G & 0 & p_1p_3(A_{13}+A_{55}) \\ 0 & p_1^2A_{66}+p_3^2A_{55}-G & 0 \\ p_1p_3(A_{13}+A_{55}) & 0 & p_1^2A_{55}+p_3^2A_{33}-G \end{pmatrix} = 0$$

The solution of (1.12) yields the three roots

$$(2.2)\qquad \begin{cases} G_1 = \frac{1}{2}\{K + \sqrt{K^2 - L^2}\} \\ G_2 = \frac{1}{2}\{K - \sqrt{K^2 - L^2}\} \\ G_3 = p_1^2A_{66} + p_3^2A_{55} \end{cases}$$

where

$$K = (A_{11}+A_{55})p_1^2 + (A_{33}+A_{55})p_3^2$$

$$L = (A_{11}p_1^2+A_{55})(A_{55}p_1^2+A_{33}p_3^2)-(A_{13}+A_{55})^2p_1^2p_3^2$$

$G_1 = 1$ corresponds to a quasi-compressional (P) wavefront propagating in the medium, while $G_2 = 1$ corresponds to a quasi-shear (SV) wavefront and $G_3 = 1$ corresponds to a quasi-shear (SH) wavefront.

The normal velocities V_P and V_{SV} of the P and SV type wavefronts can be obtained from (2.2) with $G_2 = 1$, s = 1,2, using

$$(2.3)\qquad p_1 = \frac{\sin\theta}{V_P} \qquad\qquad p_3 = \frac{\cos\theta}{V_P}$$

for the quasi-compressional case and

(2.4) $$p_1 = \frac{\sin\theta}{V_{SV}} \qquad p_3 = \frac{\cos\theta}{V_{SV}}$$

for the quasi-shear case, θ being the angle between the normal to the particular wavefront at some time and the z axis. The normal velocities are given by

(2.5) $$2V_P^2 = A_{33} + A_{55} + (A_{11}-A_{33})y + Q_y$$

(2.6) $$2V_{SV}^2 = A_{33} + A_{55} + (A_{11}-A_{33})y - Q_y$$

$$y = \sin^2\theta$$

$$Q_y = +\{(A_{33}-A_{55})^2 + 2A_1y + A_2y^2\}^{1/2}$$

$$A_1 = 2(A_{13}+A_{55})^2 - (A_{33}-A_{55})(A_{11}+A_{33}-2A_{55})$$

$$A_2 = (A_{11}+A_{33}-2A_{55})^2 - 4(A_{13}+A_{55})^2 \quad .$$

The corresponding normalized eigenvectors obtained from the eigenvalue problem are

(2.7) $$\vec{q}_P = (\ell \sin\theta\ , -m\cos\theta)$$

(2.8) $$\vec{q}_{SV} = (m\cos\theta\ , \ell\sin\theta)$$

where

$$\ell = \left\{\frac{(\Omega_y - A_{33}+A_{55})/\sin^2\theta \; + \; A_{11}+A_{33}-2A_{55}}{\Omega_y}\right\}^{\frac{1}{2}}$$

$$m = \left\{\frac{(\Omega_y - A_{11}+A_{55})/\cos^2\theta \; + \; A_{11}+A_{33}-2A_{55}}{\Omega_y}\right\}^{\frac{1}{2}}$$

and Ω_y is as above.

CHAPTER 3

REFLECTION AND TRANSMISSION COEFFICIENTS AT AN INTERFACE OF TWO TRANSVERSELY ISOTROPIC MEDIA

3.1 Introduction

In this section a general set of linear equations using asymptotic ray theory to calculate reflection and transmission coefficients in the case of a curved interface, between two transversely isotropic inhomogeneous media in welded contact will be developed.

Let $\sum$ be an interface of first order, and let $\sum$ be analytic. Denote the point at which an arbitrary wavefront strikes the interface $\sum$ by 0. The plane of incidence will be defined as the plane defined by the tangent to the wavefront normal and by the normal to the interface at the point 0 (see Fig. A1). It is assumed that the incident wavefront is propagating in the upper medium (medium 1).

Some Notation:

The displacement vectors arriving at or leaving a boundary are specified by

$$\vec{u}_v(x,y,z,t) = \sum_{n=0}^{\infty} \vec{A}_v^{(n)}(x,y,z)s_n(t-\tau_v)$$

where the index v specifies the phase and medium

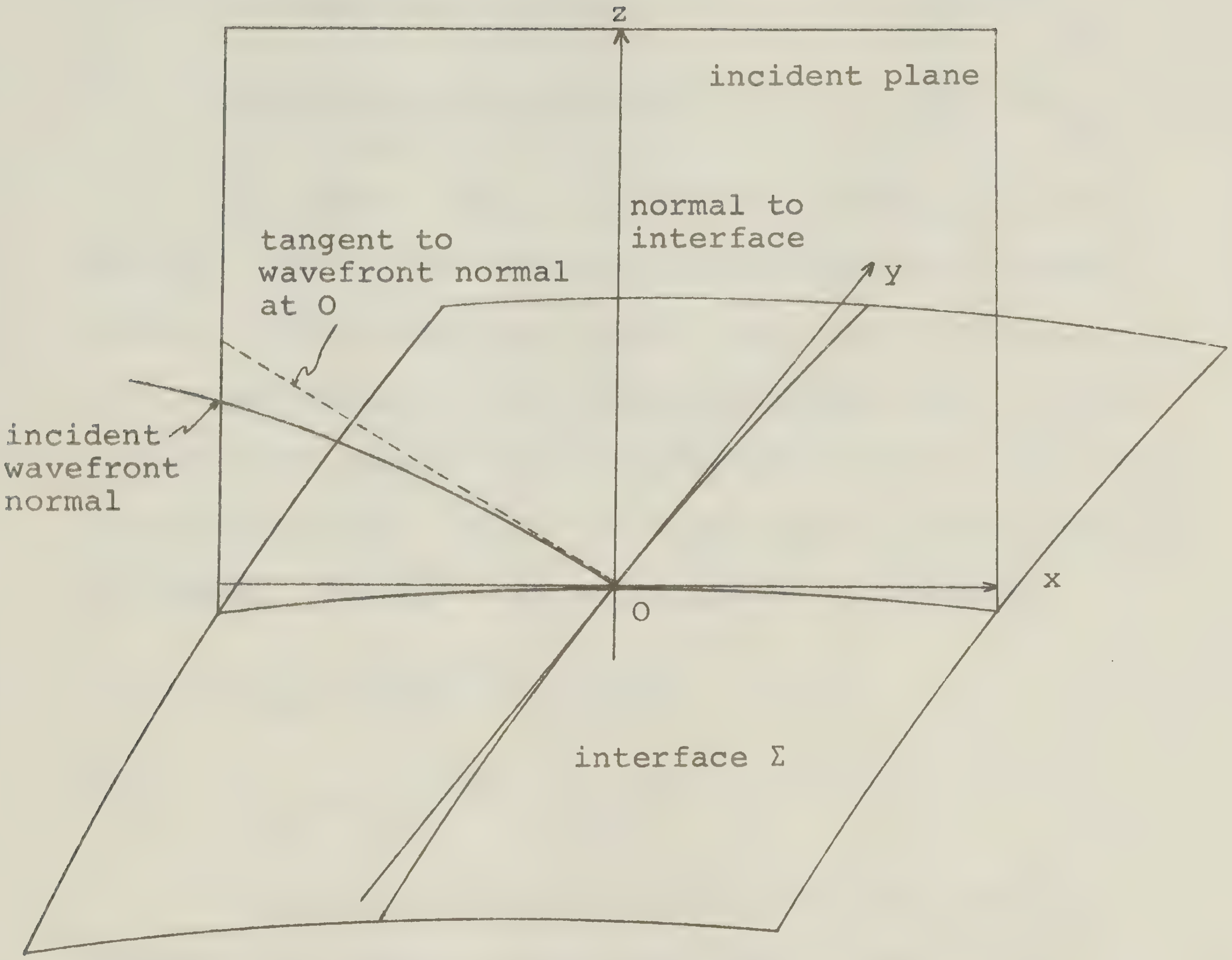


Fig. A1 Geometry of incidence.

$$(3.1)\quad \begin{cases} v=0 \text{ incident wave (either P or SV)} & \vec{A}_o^{(n)} = \vec{P}_{no} \text{ or } \vec{S}_{no} \\ v=1 \text{ reflected P wave} & \vec{A}_1^{(n)} = \vec{P}_{n1} \\ v=2 \text{ transmitted P wave} & \vec{A}_2^{(n)} = \vec{P}_{n2} \\ v=3 \text{ reflected SV wave} & \vec{A}_3^{(n)} = \vec{P}_{n3} \\ v=4 \text{ transmitted SV wave} & \vec{A}_4^{(n)} = \vec{S}_{n4} \end{cases}$$

In general, for a transversely isotropic medium, the $\vec{A}_v^{(n)}$ are neither strictly parallel to, nor perpendicular to the wavefront normal, but can be expressed as a combination of two scalar amplitudes and two unit vectors, one parallel to the wavefront normal ($\vec{g}_v^{(1)}$) and one perpendicular ($\vec{g}_v^{(2)}$) to it and lying in the plane of incidence. The $\vec{A}_v^{(n)}$ can be expressed as follows:

$$(3.2)\qquad \vec{P}_{nv} = P_{nv}\vec{g}_v^{(1)} + p_{nv}\vec{g}_v^{(2)}$$

$$(3.3)\qquad \vec{S}_{nv} = S_{nv}\vec{g}_v^{(1)} + s_{nv}\vec{g}_v^{(2)} \quad .$$

At the interface between the two media it is assumed that certain conditions must be satisfied. Since $t = \tau(x,y,z)$ on the wavefront, and time is continuous, then it must hold that τ is continuous at the interface

i.e. $\quad \tau_o(x,y,z) = \tau_v(x,y,z) \qquad v = 1,2,3,4$

and as a result the components of the slowness vector

$p_1 = \partial\tau/\partial x$, $p_2 = 0$, $p_3 = \partial\tau/\partial z$ must be continuous at the interface. From the definition of p_1 the following condition results

$$(3.4) \qquad \frac{\partial\tau_o}{\partial x} = \frac{\partial\tau_v}{\partial x} = \frac{\sin\,\theta_v}{V_v} \quad ,$$

the v being defined as in (3.1) and the θ_v are the acute angles between the wavefront normal and the z axis. Similarly

$$(3.5) \qquad \frac{\partial\tau_o}{\partial y} = \frac{\partial\tau_v}{\partial y} = 0 \quad .$$

For wavefronts propagating from the interface into the first medium, $\partial\tau_v/\partial z > 0$ for $v = 1,3$ at 0. Similarly $\partial\tau_v/\partial z < 0$ for $v = 1,2,4$ at 0. Then at point 0,

$$(3.6) \qquad \frac{\partial\tau_v}{\partial z} = (-1)^{v+1}\,\frac{\cos\,\theta_v}{V_v} \quad .$$

If $\partial\tau_v/\partial x > 1/V_v$ then $\sin\,\theta_v > 1$ and by the relation

$$p_1^2 + p_3^2 = \nabla\tau_v^2 = \left(\frac{\partial\tau_v}{\partial x}\right) + \left(\frac{\partial\tau_v}{\partial z}\right) = \frac{1}{V_v^2} \quad ,$$

it follows that

$$\cos^2\theta_v = \pm i(\sin^2\theta_v - 1)^{1/2} \quad .$$

The choice of sign is determined by the physical condition that the solution for

$$s_n(t-\tau_v) = \frac{1}{\pi}\,\mathrm{Re}\int_{\omega_o}^{\infty} \frac{\tilde{S}(\omega)}{(i\omega)^n}\, e^{i\omega(t-\tau_v)}\, d\omega$$

approach zero as the wavefront moves away from the interface. Thus the minus sign is chosen.

3.2 Incident Quasi-compressional P Wavefront

As shown in Fig. A2, the solid lines indicate normals to the P-type wavefronts, while the dashed lines define SV type wavefront normals. This notation will be used throughout the thesis. The eigenvectors $\vec{g}_v^{(i)}$ are given in terms of components in the Cartesian system as

$$\vec{g}_v^{(1)} = (\ell_v \sin\theta_v,\ (-1)^{v+1} m_v \cos\theta_v)$$

$$\vec{g}_v^{(2)} = (m_v \cos\theta_v,\ (-1)^{v}\ \ell_v \sin\theta_v)$$

$$\ell_v = \left\{\frac{[(Q^{(v)} - A_{33}^{(v)} + A_{55}^{(v)})/\sin^2\theta_v] + [A_{11}^{(v)} + A_{33}^{(v)} - 2A_{55}^{(v)}]}{Q^{(v)}}\right\}^{\frac{1}{2}}$$

$$m_v = \left\{\frac{[(Q^{(v)} - A_{11}^{(v)} + A_{55}^{(v)})/\cos^2\theta_v] + [A_{11}^{(v)} + A_{33}^{(v)} - 2A_{55}^{(v)}]}{Q^{(v)}}\right\}^{\frac{1}{2}}$$

$$Q^{(v)} = \{(A_{33}^{(v)} - A_{55}^{(v)})^2 + 2\,A_1^{(v)} \sin\theta_v + A_2^{(v)} \sin^2\theta_v\}^{\frac{1}{2}}$$

$$A_1^{(v)} = 2(A_{13}^{(v)} + A_{55}^{(v)})^2 - (A_{33}^{(v)} - A_{55}^{(v)})(A_{11}^{(v)} + A_{33}^{(v)} - 2A_{55}^{(v)})$$

$$A_2^{(v)} = (A_{11}^{(v)} + A_{33}^{(v)} - 2A_{55}^{(v)})^2 - 4(A_{13}^{(v)} + A_{55}^{(v)})^2$$

A_{ij} - defined in Chapter 2.

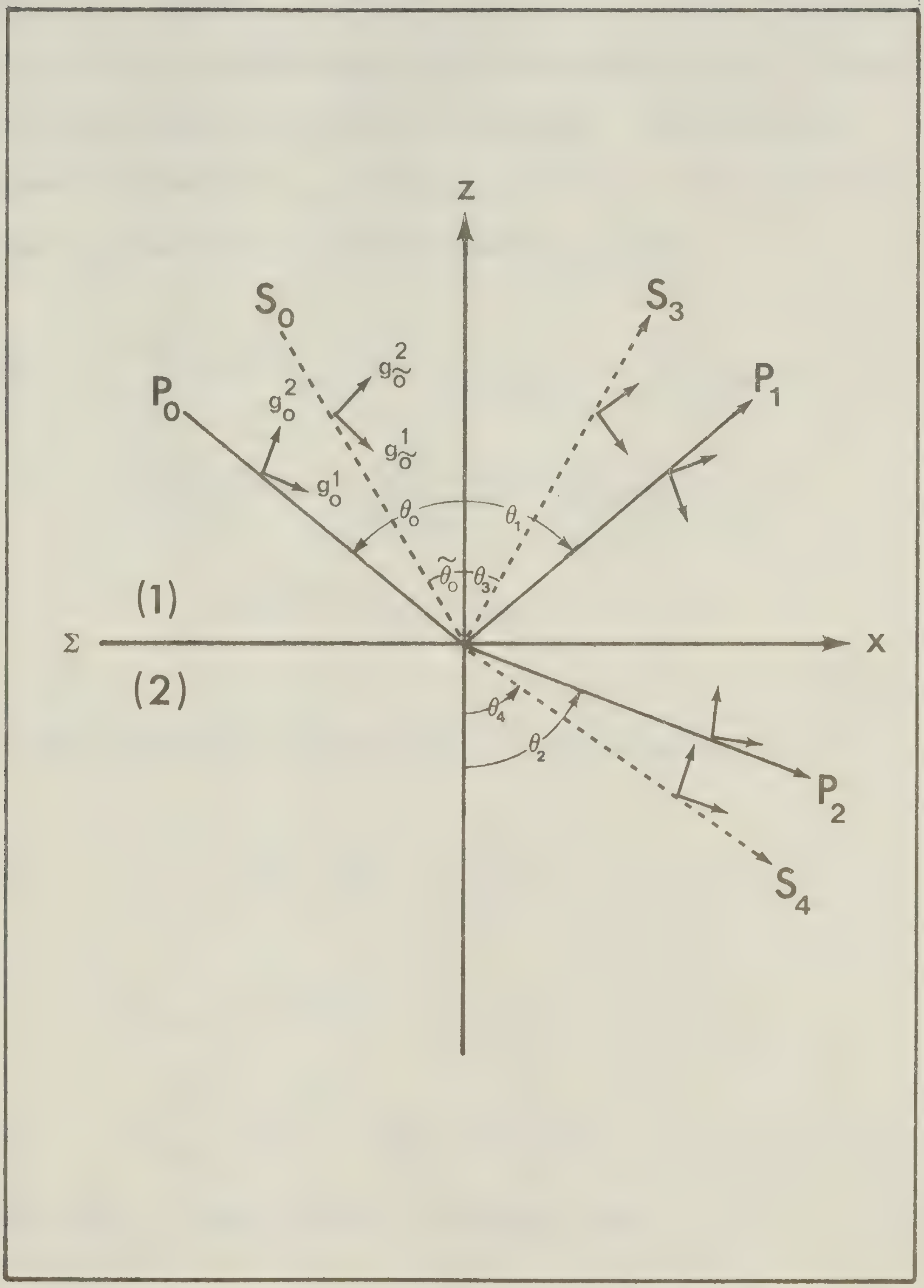


Fig. A2 Orientation of wavefront normals and eigenvectors for an incident quasi-compressional P or quasi-shear SV wavefront.

The conditions which must hold at the interface between two layers in welded contact and the continuity of the x and z components of displacement, and the continuity of normal and shear stresses. Introducing Kroneker's symbol (δ_{ij}: $=0, i \neq j$; $=1, i=j$) the continuity of the x component of displacement yields

$$\text{(3.7)} \qquad \sum_{v=1}^{2} (-1)^{v} P_{nv} \ell_{v} \sin\theta_{v} + \sum_{v=3}^{4} (-1)^{v} S_{nv} m_{v} \cos\theta_{v} =$$

$$P_{no} \ell_{o} \sin\theta_{o} - \sum_{v=3}^{4} (-1)^{v} S_{nv} \ell_{v} \sin\theta_{v}$$

$$- \sum_{v=o}^{2} (-1)^{\delta_{vo}+v} P_{nv} m_{v} \cos\theta_{v}$$

Continuity of the z component of displacement has

$$\text{(3.8)} \qquad \sum_{v=1}^{2} P_{nv}\, m_{v} \cos\theta_{v} - \sum_{v=3}^{4} S_{nv}\, \ell_{v} \sin\theta_{v} =$$

$$P_{no}\, m_{o} \cos\theta_{o} - \sum_{v=3}^{4} S_{nv}\, m_{v} \cos\theta_{v}$$

$$+ \sum_{v=o}^{2} (-1)^{\delta_{vo}} P_{nv}\, \ell_{v} \sin\theta_{v}$$

Continuity of shear stress requires that

$$\sum_{v=1}^{4} (-1)^{v}\, \sigma_{xz}(\vec{u}_{v}) = \sigma_{xz}(\vec{u}_{o})$$

where

$$\sigma_{xz}(\vec{u}_v) = C_{55}^{(v)} \left(\frac{\partial (u_x)_v}{\partial z} + \frac{\partial (u_z)_v}{\partial x} \right)$$

and

$$C_{ij}^{(v)} = \rho^{(v)} A_{ij}^{(v)}$$

The superscript on the C_{55} identifies the medium. At the interface $C_{55}^{(o)} = C_{55}^{(1)} = C_{55}^{(3)}$ and $C_{55}^{(2)} = C_{55}^{(4)}$.

After some algebra the following expression results:

$$\sum_{v=1}^{2} \frac{C_{55}^{(v)} (\ell_v + m_v) \sin 2\theta_v P_{nv}}{V_v} + \sum_{v=3}^{4} \frac{C_{55}^{(v)} (m_v \cos^2\theta_v - \ell_v \sin^2\theta_v) s_{nv}}{V_v}$$

$$(3.9) \qquad = \frac{C_{55}^{(o)} (\ell_o + m_o) \sin 2\theta_o P_{no}}{V_o}$$

$$+ \sum_{v=0}^{2} \left\{ \frac{(-1)^{\delta_{vo}+1} C_{55}^{(v)} (m_v \cos^2\theta_v - \ell_v \sin^2\theta_v) p_{nv}}{V_v} \right.$$

$$\left. - (-1)^{\delta_{vo}+v} C_{55}^{(v)} \left(\frac{\partial (P_x)_{n-1,v}}{\partial z} + \frac{\partial (P_z)_{n-1,v}}{\partial x} \right) \right\}$$

$$- \sum_{v=3}^{4} \left\{ \frac{C_{55}^{(v)} (\ell_v + m_v) \sin 2\theta_v S_{nv}}{V_v} \right.$$

$$+ (-1)^{v} C_{55}^{(v)} \left(\frac{\partial (S_x)_{n-1,v}}{\partial z} + \frac{\partial (S_x)_{n-1,v}}{\partial x} \right) \Bigg\}$$

Continuity of normal stress at the interface requires that

$$\sum_{v=1}^{4} (-1)^{v} \sigma_{zz}(\vec{u}_v) = \sigma_{zz}(\vec{u}_o)$$

with

$$\sigma_{zz}(\vec{u}_v) = C_{13}^{(v)} \frac{\partial (u_x)_v}{\partial x} + C_{33}^{(v)} \frac{\partial (u_z)_v}{\partial z}$$

This yields the condition

$$(3.10) \quad \sum_{v=1}^{2} (-1)^{v} \frac{[\ell_v C_{13}^{(v)} + (m_v C_{33}^{(v)} - \ell_v C_{13}^{(v)}) \cos^2\theta_v] \, P_{nv}}{V_v}$$

$$- \sum_{v=3}^{4} (-1)^{v} \frac{(\ell_v C_{33}^{(v)} - m_v C_{13}^{(v)}) \sin 2\theta_v \, s_{nv}}{V_v}$$

$$= \frac{[\ell_o C_{13}^{(o)} + (m_o C_{33}^{(o)} - \ell_o C_{13}^{(o)}) \cos^2\theta_o] \, P_{no}}{V_o}$$

$$+ \sum_{v=0}^{2} (-1)^{\delta_{vo}+v} \Bigg\{ C_{13}^{(v)} \frac{\partial (P_x)_{n-1,v}}{\partial x} + \frac{(\ell_v C_{33}^{(v)} - m_v C_{13}^{(v)}) \sin 2\theta_v \, p_{nv}}{V_v}$$

$$+ C_{33}^{(v)} \frac{\partial (P_z)_{n-1,v}}{\partial z} \Bigg\} + \sum_{v=3}^{4} (-1)^{v} \Bigg\{ C_{13}^{(v)} \frac{\partial (S_x)_{n-1,v}}{\partial x} + \frac{\partial (S_z)_{n-1,v}}{\partial z}$$

$$- \frac{[\ell_v C_{13}^{(v)} + (m_v C_{33}^{(v)} - \ell_v C_{13}^{(v)}) \cos^2 \theta_v] \, S_{nv}}{V_v} \Bigg\}$$

3.3 Incident Quasi-shear SV Wavefront

The geometry of incidence is shown in Fig. A2 and the four continuity equations obtained in an analogous manner to the preceding are:

Continuity of x component of displacement

$$(3.11) \quad \sum_{v=1}^{2} (-1)^v P_{nv} \ell_v \sin \theta_v + \sum_{v=3}^{4} (-1)^v s_{nv} m_v \cos \theta_v =$$

$$s_{no} \tilde{m}_o \cos \tilde{\theta}_o - \sum_{v=1}^{2} (-1)^v p_{nv} m_v \cos \theta_v$$

$$+ \sum_{\substack{v=3 \\ v=0}}^{4} (-1)^{\delta_{vo}+v} S_{nv} \ell_v \sin \theta_v$$

Continuity of the z component of displacement

$$(3.12) \quad \sum_{v=3}^{4} s_{nv} \ell_v \sin \theta_v - \sum_{v=1}^{2} P_{nv} m_v \cos \theta_v =$$

$$s_{no} \tilde{\ell}_o \sin \tilde{\theta}_o - \sum_{v=1}^{2} p_{nv} \ell_v \sin \theta_v$$

$$+ \sum_{\substack{v=3 \\ v=o}}^{4} (-1)^{\delta_{vo}} S_{nv} m_v \cos \theta_v$$

Continuity of shear stress

$$\sum_{v=1}^{2} \frac{C_{55}^{(v)} (\ell_v+m_v)\sin 2\theta_v P_{nv}}{V_v} + \sum_{v=3}^{4} \frac{C_{55}^{(v)} (m_v\cos^2\theta_v-\ell_v\sin^2\theta_v) s_{nv}}{V_v}$$

$$(3.13) \qquad = \frac{C_{55}^{(o)} (\tilde{m}_o\cos^2\tilde{\theta}_o-\tilde{\ell}_o\sin^2\theta_o) s_{no}}{V_v}$$

$$- \sum_{v=1}^{2} \frac{C_{55}^{(v)} (m_v\cos^2\theta_v - \ell_v\sin^2\theta_v) p_{nv}}{V_v}$$

$$+ (-1)^v C_{55}^{(v)} \left(\frac{\partial (P_x)_{n-1,v}}{\partial z} + \frac{\partial (P_z)_{n-1,v}}{\partial x} \right)$$

$$+ \sum_{\substack{v=3 \\ v=0}}^{4} (-1)^{\delta_{vo}+1} \frac{C_{55}^{(v)} (\ell_v+m_v)\sin 2\theta_v S_{nv}}{V_v}$$

$$+ (-1)^{\delta_{vo}+v} C_{55}^{(v)} \left(\frac{\partial (S_x)_{n-1,v}}{\partial z} + \frac{\partial (S_z)_{n-1,v}}{\partial x} \right)$$

Continuity of normal stress

$$\sum_{v=1}^{2} (-1)^{v+1} \frac{[\ell_v C_{13}^{(v)} + (m_v C_{33}^{(v)} - \ell_v C_{13}^{(v)})\cos^2\theta_v] P_{nv}}{V_v}$$

$$(3.14) \qquad + \sum_{v=3}^{4} (-1)^v \frac{(\ell_v C_{33}^{(v)} - m_v C_{13}^{(v)})\sin 2\theta_v s_{nv}}{2V_v}$$

$$= \frac{(\tilde{\ell}_o C_{33}^{(o)} - \tilde{m}_o C_{13}^{(o)}) \sin 2\tilde{\theta}_o s_{no}}{2\tilde{V}_o} +$$

$$+\sum_{v=1}^{2}(-1)^{v+1}\left\{C_{13}^{(v)}\ \frac{\partial (P_x)_{n-1,v}}{\partial x}+\frac{(\ell_v C_{33}^{(v)}-m_v C_{13}^{(v)})\sin 2\theta_v\ p_{nv}}{2V_v}\right.$$

$$\left.+\ C_{33}^{(v)}\ \frac{\partial (P_z)_{n-1,v}}{\partial z}\right\}-\sum_{\substack{v=3\\v=0}}^{4}(-1)^{\delta_{vo}+v}\left\{C_{13}^{(v)}\ \frac{\partial (S_x)_{n-1,v}}{\partial x}\right.$$

$$\left.+\ C_{33}^{(v)}\ \frac{\partial (S_z)_{n-1,v}}{\partial z}-\frac{[\ell_v C_{13}^{(v)}+(m_v C_{33}^{(v)}-\ell_v C_{13}^{(v)})\cos^2\theta_v]S_{nv}}{V_v}\right\}$$

where $\tilde{V}_o$ denotes the normal velocity of the incident SV wavefront.

The equations (3.7), (3.8), (3.9), (3.10) and (3.11), (3.12), (3.13), (3.14) are two recursive sets of four linear equations in four unknowns. To know the solution of these two sets for a given n, the solutions for all m<n must be known.

If the geometrical parameters of the layers are much greater than the wavelength, and if the observer is not very close to the source of the wave, taking only the first term in the asymptotic series (n=0) can be justified to give a reasonable approximation. This so called zero order or plane wave approximation will be considered.

As a further simplification, it will be assumed that the axes of anisotropy in both media are perpendicular to the interface. Thus, for an incident P type wavefront $\theta_o = \theta_1$, and for an incident SV type wavefront $\tilde{\theta}_o = \theta_3$.

Let $\sin \theta_1 = x$. All other pertinent angles will be parameterized in this variable via Snell's Law.

$$\left.\begin{array}{c} \dfrac{\sin \theta_o}{V_o} \\[2em] \dfrac{\sin \tilde{\theta}_o}{\tilde{V}_o} \end{array}\right\} = \frac{\sin \theta_v}{V_v} \qquad v = 1,2,3,4.$$

Also let

$$\frac{V_1}{V_2} = n \quad , \quad \frac{V_4}{V_2} = k_2 \quad , \quad \frac{V_3}{V_1} = k_1 .$$

Using these above formulae, the following simplifying substitutions will be carried out:

$$\cos \theta_1 = P = (1 - x^2)^{\frac{1}{2}}$$

$$\cos \theta_2 = Q = (1 - k_1^2 x^2)^{\frac{1}{2}}$$

$$\cos \theta_3 = S = (1 - \frac{x^2}{n^2})^{\frac{1}{2}}$$

$$\cos \theta_4 = R = (1 - \frac{k_2^2 x^2}{n^2})^{\frac{1}{2}}$$

$$C_{55}^{(v)} = \begin{cases} \beta_1 & v = 0,1,3. \\ \beta_2 & v = 2,4. \end{cases}$$

$$\ell_3 C_{33}^{(3)} - m_3 C_{13}^{(3)} = \delta_1$$

$$\ell_4 C_{33}^{(4)} - m_4 C_{13}^{(4)} = \delta_2$$

$$\frac{V_1}{V_3}\,\frac{1}{(\ell_1+m_1)}\,(m_4\cos^2\theta_3 - \ell_3\sin^2\theta_3) = \omega_1$$

$$\frac{V_1}{V_4}\,\frac{1}{(\ell_1+m_1)}\,(m_4\cos^2\theta_4 - \ell_4\sin^2\theta_4) = \omega_2$$

$$\ell_1 C_{13}^{(1)} + (m_1 C_{33}^{(1)} - \ell_1 C_{13}^{(1)})\cos^2\theta_1 = \varepsilon_1$$

$$\frac{V_1}{V_2}\,[\ell_2 C_{13}^{(2)} + (m_2 C_{33}^{(2)} - \ell_2 C_{13}^{(2)})\cos^2\theta_2] = \varepsilon_2$$

$$\frac{\ell_2 + m_2}{\ell_1 + m_1} = \ell$$

After implementing the substitutions in the zero order equations, the two following sets of four linear equations in four unknowns result. The unknowns are the amplitudes of the reflected and refracted disturbances, as the amplitude of the incident disturbance is assumed known.

Incident P type wavefront

(3.15)

$$\begin{bmatrix} x & \frac{m_3}{\ell_1} Q & -\frac{\ell_2}{\ell_1} \frac{x}{n} & -\frac{m_4}{\ell_1} R \\ P & \frac{\ell_3}{m_1} k_1 x & \frac{m_2}{m_1} S & \frac{-\ell_4 k_2 x}{m_1 n} \\ \beta_1 xP & \beta_1 \omega_1 & \beta_2 \ell x S & \beta_2 \omega_2 \\ -\varepsilon_1 & \varepsilon_1 x Q & \varepsilon_2 & \delta_2 x R \end{bmatrix} \begin{bmatrix} P_{01} \\ S_{03} \\ P_{02} \\ S_{04} \end{bmatrix} = \begin{bmatrix} -x \\ P \\ \beta_1 xP \\ \varepsilon_1 \end{bmatrix} P_{00}$$

Incident SV type wavefront

(3.16)

$$\begin{bmatrix} x & \frac{m_3}{\ell_1} Q & -\frac{\ell_2}{\ell_1} \frac{x}{n} & -\frac{m_4}{\ell_1} R \\ P & \frac{\ell_3}{m_1} k_1 x & \frac{m_2}{m_1} S & \frac{-\ell_4 k_2 x}{m_1 n} \\ \beta_1 xP & \beta_1 \omega_1 & \beta_2 \ell x S & \beta_2 \omega_2 \\ -\varepsilon_1 & \delta_1 x Q & \varepsilon_2 & -\delta_2 x R \end{bmatrix} \begin{bmatrix} P_{01} \\ S_{03} \\ P_{02} \\ S_{04} \end{bmatrix} = \begin{bmatrix} -\frac{m_3}{\ell_1} Q \\ -\frac{\ell_3}{m_1} k_1 x \\ \beta_1 \omega_1 \\ -\delta_1 x Q \end{bmatrix} S_{00}$$

Solving (3.15) and (3.16) using Cramer's method with the incident displacement set to unity the reflection and transmission coefficient $R_{\mu\nu}$ tabulated below are obtained. (According to our convention, $\mu = 1$ corresponds to an incident P wavefront and $\mu = 3$ corresponds to an incident SV wavefront, while the second subscript ν typifies the resultant reflected or transmitted wavefront.)

$$
(3.17) \quad \left\{
\begin{aligned}
R_{11} &= (-E_1+E_2+E_3+E_4-E_5-E_6)/D \\
R_{12} &= (E_7+E_8)/D \\
R_{13} &= (E_9+E_{10})/D \\
R_{14} &= (E_{11}+E_{12})/D \\
R_{31} &= (E_{13}+E_{14})/D \\
R_{32} &= (E_{15}+E_{16})/D \\
R_{33} &= (-E_1+E_2-E_3+E_4+E_5-E_6)/D \\
R_{34} &= (E_{17}+E_{18})/D \\
D &= E_1+E_2+E_3+E_4+E_5+E_6
\end{aligned}
\right.
$$

The E_j , $j = 1, 18.$ are given explicitly in the appendix.

3.4 Free Interface

The free interface case has a wavefront (either quasi-shear SV or quasi-compressional P) propagating in a transversely isotropic medium and impinging on the interface of that medium and a vacuum. Reflected shear SV and compressional waves only result.

The boundary requirements are the continuity of shear and normal stresses along with the previously discussed conditions on the first derivatives of τ_v (see Fig. A3).

$$\vec{g}_o^{(1)} = (\ell_o \sin\theta_o,\ m_o \cos\theta_o) \qquad \vec{g}_{\tilde{o}}^{(1)} = (\tilde{\ell}_o \sin\tilde{\theta}_o,\ \tilde{m}_o \cos\tilde{\theta}_o)$$

$$\vec{g}_o^{(2)} = (-m_o \cos\theta_o,\ \ell_o \sin\theta_o) \qquad \vec{g}_{\tilde{o}}^{(2)} = (-\tilde{m}_o \cos\tilde{\theta}_o,\ \tilde{\ell}_o \sin\tilde{\theta}_o)$$

$$\vec{g}_2^{(1)} = (\ell_2 \sin\theta_2,\ -m_2 \cos\theta_2) \qquad \vec{g}_4^{(1)} = (\ell_4 \sin\theta_4,\ -m_4 \cos\theta_4)$$

$$\vec{g}_2^{(2)} = (-m_2 \cos\theta_2,\ -\ell_2 \sin\theta_2) \qquad \vec{g}_4^{(2)} = (-m_4 \cos\theta_4,\ -\ell_4 \sin\theta_4)$$

The requirement for the continuity of shear stress $\sigma_{xz}(\vec{u}_o) + \sigma_{xz}(\vec{u}_2) + \sigma_{xz}(\vec{u}_4) = 0$ yields the following two equations.

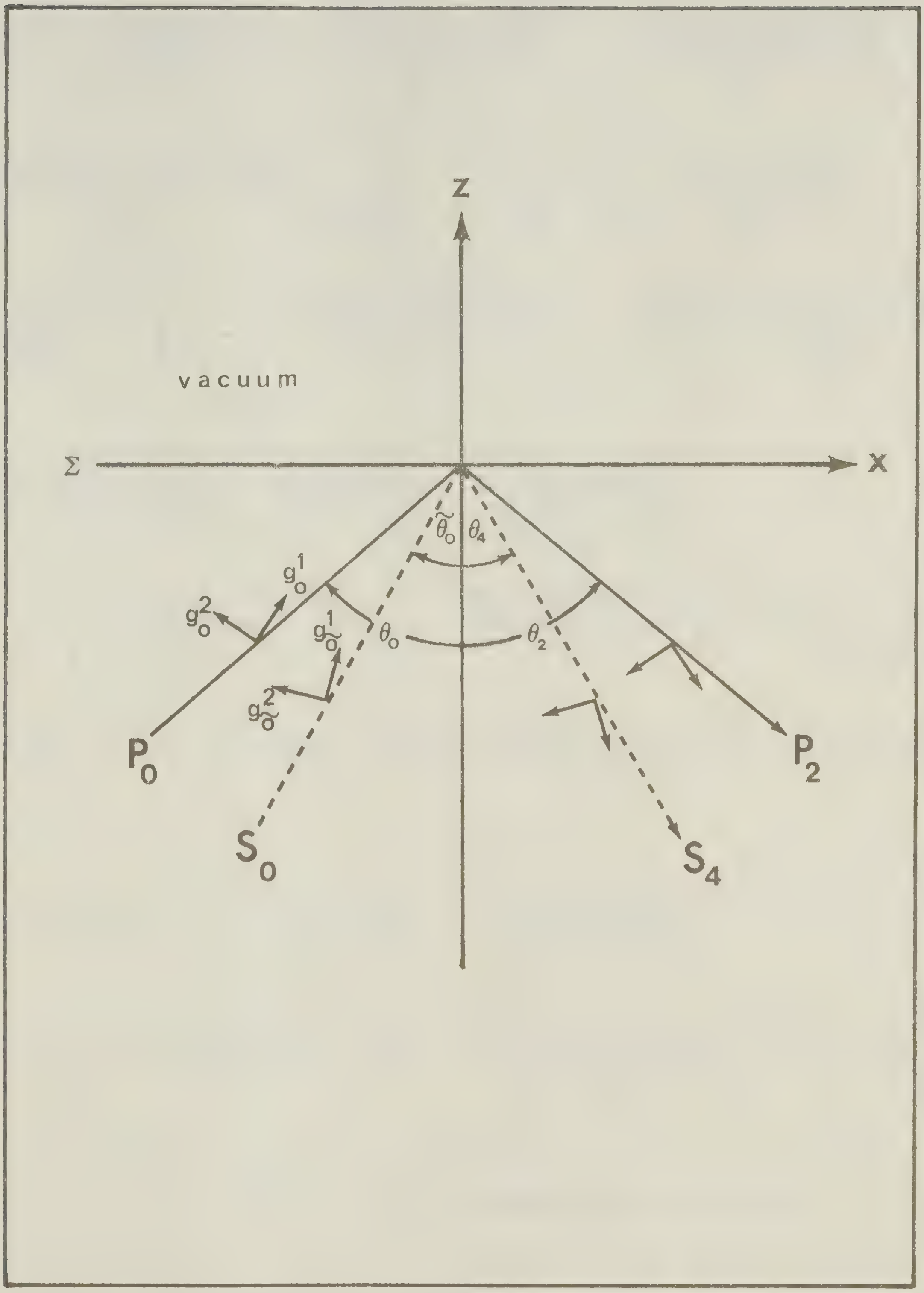


Fig. A3 Wavefront normals and eigenvectors for the free interface case.

$$F = C_{55}^{(2)} \left[\left\{ \frac{\partial (P_z)_{n-1,2}}{\partial x} + \frac{\partial (P_x)_{n-1,2}}{\partial z} \right\} + \left\{ \frac{(\ell_2+m_2)\sin 2\theta_2 P_{n2}}{V_2} \right.\right.$$

$$\left.\left. - \frac{(m_2\cos^2\theta_2-\ell_2\sin^2\theta_2)}{V_2}\, p_{n2} \right\} \right] + C_{55}^{(4)} \left[\left\{ \frac{\partial (S_z)_{n-1,4}}{\partial x} \right.\right.$$

$$\left.\left. + \frac{\partial (S_x)_{n-1,4}}{\partial z} \right\} + \left\{ \frac{(\ell_4+m_4)\sin 2\theta_4 S_{n4}}{V_4} - \frac{(m_4\cos^2\theta_4-\ell_4\sin^2\theta_4)s_{n4}}{V_4} \right\} \right]$$

where

$$F = C_{55}^{(o)} \left[- \left\{ \frac{\partial (P_z)_{n-1,o}}{\partial x} + \frac{\partial (P_x)_{n-1,o}}{\partial z} \right\} \right.$$

$$\left. + \left\{ \frac{(\ell_o+m_o)\sin 2\theta_o}{V_o} P_{no} - \frac{(m_o\cos^2\theta_o-\ell_o\sin^2\theta_o)}{V_o}\, p_{no} \right\} \right] \tag{3.18}$$

in the case of an incident quasi-compressional P wavefront and

$$F = C_{55}^{(o)} \left[- \left\{ \frac{\partial (S_z)_{n-1,\tilde{o}}}{\partial x} + \frac{\partial (S_x)_{n-1,\tilde{o}}}{\partial z} \right\} \right.$$

$$\left. + \left\{ \frac{(\tilde{\ell}_o+\tilde{m}_o)\sin 2\tilde{\theta}_o}{\tilde{V}_o} S_{no} - \frac{(\tilde{m}_o\cos^2\tilde{\theta}_o-\tilde{\ell}_o\sin^2\tilde{\theta}_o)}{\tilde{V}_o}\, s_{no} \right\} \right] \tag{3.19}$$

in the case of an incident quasi-shear SV wavefront.

Continuity of normal stress $\sigma_{zz}(\vec{u}_o) + \sigma_{zz}(\vec{u}_2) + \sigma_{zz}(\vec{u}_4) = 0$ gives rise to

$$G = \left[-\left\{ C_{13}^{(2)} \frac{\partial (P_x)_{n-1,2}}{\partial x} + C_{33}^{(2)} \frac{\partial (P_z)_{n-1,2}}{\partial z} \right\} \right.$$

$$+ \left\{ \frac{(\ell_2 C_{13}^{(2)} + (m_2 C_{33}^{(2)} - \ell_2 C_{13}^{(2)})\cos^2\theta_2) P_{n2}}{V_2} \right.$$

$$\left. \left. + \quad \frac{(\ell_2 C_{33}^{(2)} - m_2 C_{13}^{(2)})\sin 2\theta_2 p_{n2}}{2V_2} \right\} \right]$$

$$\left[-\left\{ C_{13}^{(4)} \frac{\partial (S_x)_{n-1,4}}{\partial x} + C_{33}^{(4)} \frac{\partial (S_z)_{n-1,4}}{\partial z} \right\} \right.$$

$$+ \left\{ \frac{(\ell_4 C_{13}^{(4)} + (m_4 C_{33}^{(4)} - \ell_4 C_{13}^{(4)})\cos^2\theta_4) S_{n4}}{V_4} \right.$$

$$\left. \left. + \quad \frac{(\ell_4 C_{33}^{(4)} - m_4 C_{13}^{(4)})\sin 2\theta_4 \; s_{n4}}{2V_4} \right\} \right]$$

where again

$$(3.20) \qquad G = \left[\left\{ \frac{C_{13}^{(o)} \partial (P_x)_{n-1,o}}{\partial x} + \frac{C_{33}^{(o)} \partial (P_z)_{n-1,o}}{\partial z} \right\} \right.$$

$$- \left\{ \frac{(\ell_o C_{13}^{(o)} + (m_o C_{13}^{(o)} - \ell_o C_{13}^{(o)})\cos^2\theta_o) P_{no}}{V_o} \right.$$

$$\left. \left. + \quad \frac{(\ell_o C_{33}^{(o)} - m_o C_{13}^{(o)} \sin 2\theta_o \; p_{no}}{2V_o} \right\} \right]$$

corresponds to an incident P wavefront and

$$(3.21) \qquad G = \left[\left\{ \frac{C_{13}^{(o)} \partial (S_x)_{n-1,\tilde{o}}}{\partial x} + \frac{C_{33}^{(o)} \partial (S_z)_{n-1,\tilde{o}}}{\partial z} \right\} \right.$$

$$- \left\{ \frac{(\tilde{\ell}_o C_{13}^{(o)} + (\tilde{m}_o C_{33}^{(o)} - \tilde{\ell}_o C_{13}^{(o)}) \cos^2 \tilde{\theta}_o) S_{no}}{\tilde{V}_o} \right.$$

$$\left. + \quad \frac{(\tilde{\ell}_o C_{33}^{(o)} - \tilde{m}_o C_{13}^{(o)}) \sin 2\tilde{\theta}_o \ s_{no}}{2\tilde{V}_o} \right\} \Bigg]$$

corresponds to an incident SV wavefront.

Equations (3.18), (3.19) and (3.20), (3.21) are two sets of two recursive linear equations in two unknowns whose solutions give the required reflection coefficients.

In the zero order approximation (again assuming the axis of anisotropy is perpendicular to the interface) these systems become (3.22) and (3.23) after the following simplifying substitutions.

$$x = \sin \theta_2 \qquad\qquad k_1 = V_4/V_2$$

$$\cos \theta_2 = P \qquad\qquad \cos \theta_4 = Q$$

$$\delta_1 = \ell_4 C_{33}^{(4)} - m_4 C_{13}^{(4)}$$

$$\omega_1 = \frac{V_2}{V_4} \frac{1}{(\ell_2 + m_2)} (m_4 \cos^2 \theta_4 - \ell_4 \sin^2 \theta_4)$$

$$\varepsilon_1 = \ell_2 C_{13}^{(2)} + (m_2 C_{33}^{(2)} - \ell_2 C_{13}^{(2)}) \cos^2 \theta_2$$

Incident P wavefront

$$(3.22) \qquad \begin{bmatrix} xP & -\omega_1 \\ \varepsilon_1 & \delta_1 xQ \end{bmatrix} \begin{bmatrix} P_{02} \\ s_{04} \end{bmatrix} = \begin{bmatrix} xP \\ -\varepsilon_1 \end{bmatrix}$$

Incident SV wavefront

$$(3.23)\qquad \begin{bmatrix} xP & -\omega_1 \\ \varepsilon_1 & \delta_1 xQ \end{bmatrix} \begin{bmatrix} P_{02} \\ s_{04} \end{bmatrix} = \begin{bmatrix} -\omega_1 \\ -\delta_1 xQ \end{bmatrix} s_{oo}$$

Solving (3.22) and (3.23) yields the following coefficients of reflection.

$$(3.24)\qquad \begin{cases} R_{o2} = (\delta_1 x^2 PQ - \varepsilon_1\omega_1)/H \\ R_{o4} = -2\varepsilon_1 xP/H \\ R_{\tilde{o}4} = (-\delta_1 x^2 PQ + \varepsilon_1\omega_1)/H \\ R_{\tilde{o}2} = -2\delta_1\omega_1 xQ/H \\ H \quad = \delta_1 x^2 PQ + \varepsilon_1\omega_1 \end{cases}$$

3.5 Surface Conversion Coefficients

A receiver situated on the earth's surface i.e. at the interface of a transversely isotropic medium and a vacuum, records not only the disturbance caused by an incident wavefront at that point, but also the disturbances resulting from the two reflected wavefronts.

Let Σ denote the interface, G the location of the receiver at the interface and $\vec{u}_v^{\Sigma}(G)$ the displacement registered by the receiver. ($v = 0$ or $\tilde{o}$ depending on whether the incident wavefront is of the P or SV type.) Thus

$$(3.25) \qquad \vec{u}_v^{\Sigma}(G) = \vec{u}_v(G) + \vec{u}_{v2}(G) + \vec{u}_{v4}(G)$$

where the first term on the R.H.S. of (3.25) is the contribution from the incident wavefront and the second and third terms are contributions from the reflected P and SV wavefronts respectively.

In the zero order approximation in asymptotic ray theory an arbitrary displacement vector can be defined by

$$\vec{u}_{\mu}(G) = A_{\mu}(G)\exp i\omega(t-\tau_{\mu}(G))\vec{n}_{\mu}(G)$$

$\vec{n}_{\mu}(G)$ being a unit vector.

As $\tau_o(G) = \tau_2(G) = \tau_4(G)$ at the interface, for an incident P wavefront (3.25) becomes

$$(3.26) \qquad \vec{u}_o^{\Sigma}(G) = P_{oo}(G) \exp i\omega(t-\tau_o(G))\{\vec{n}_o(G)$$

$$+ R_{o2}\vec{n}_2(G) + R_{o4}\vec{n}_4(G)\}$$

and similarly since $\tau_{\tilde{o}}(G) = \tau_2(G) = \tau_4(G)$, the resultant expression for an incident SV wavefront is

$$\text{(3.27)} \qquad \vec{u}_o^{\Sigma}(G) = S_{oo}(G) \exp i\omega\, (t-\tau_{\tilde{o}}(G))\; \vec{n}_{\tilde{o}}(G)$$

$$+ R_{\tilde{o}2}\vec{n}_2(G) + R_{\tilde{o}4}\vec{n}_4(G)\} \;.$$

The $R_{\nu\mu}$ are given by (3.24).

Let $\vec{g}_\nu(G) = \vec{n}_\nu(G) + R_{\nu 2}\vec{n}_2(G) + R_{\nu 4}\vec{n}_4(G)$.

$\vec{g}_\nu(G)$ is called the surface conversion vector and the x and z components of $\vec{g}_\nu(s)$, $g_{\nu x}$ and $g_{\nu z}$, are called the surface conversion coefficients. From (3.24) it follows

$$\text{(3.28)} \quad \left\{ \begin{aligned} g_{ox} &= \frac{2x(\varepsilon_1 m_4 + \delta_1 x^2 \ell_2)PQ}{H} \\ g_{oz} &= \frac{2\varepsilon_1(\omega_1 m_2 + k_1 x^2 \ell_4)P}{H} \\ g_{\tilde{o}x} &= \frac{2\omega_1(\varepsilon_1 m_4 + \delta_1 x^2 \ell_2)Q}{H} \\ g_{\tilde{o}z} &= \frac{2\delta_1 x(\omega_1 m_2 + k_1 x^2 \ell_4)PQ}{H} \end{aligned} \right.$$

CHAPTER 4

ELLIPSOIDAL VELOCITY ANISOTROPY

4.1 Introduction

Analytic expressions for reflection and transmission coefficients at the interface of two traversely isotropic media were obtained in Chapter 3 using the zero-order approximation of asymptotic ray theory, which amounts to using a plane wave approximation. Since the expressions involving the wavefront normal velocities are still quite complicated, and as these velocities are required in calculations involving Snell's Law, a simpler model is sought to analyze here. The model which will be considered is one consisting of homogeneous plane layers which display ellipsoidal velocity anisotropy (all wavefronts are ellipsoids of revolution). The major axes of these ellipsoids will be assumed to be either parallel or perpendicular to the plane interfaces.

As was shown in Chapter 3 a transversely isotropic medium can be specified by the five modified elastic parameters A_{11}, A_{13}, A_{33}, A_{55}, and A_{66} where $A_{ij} = C_{ij/\rho}$, ρ being the density of the medium, which is constant if the medium is homogeneous and C_{ij} are the elastic parameters of the medium. Since A_{66} is inherent to SH type wavefronts, which are not considered, only 4 modified

elastic parameters need be known for the purposes of this discussion to describe the medium. The further constraint that the wavefronts are all ellipsoids of revolution reduces this number to 3, since for ellipsoidal wavefronts the condition

$$(4.1) \qquad (A_{13}+A_{55})^2-(A_{11}-A_{55})(A_{33}-A_{55}) = 0$$

holds, and as a result A_{13} can be expressed in terms of A_{11}, A_{33}, and A_{55} (see Gassmann (5)).

Upon substitution of (4.1) into (2.4) and (2.5), the equations for the normal velocities, the following expressions are obtained.

$$(4.2) \qquad V_P^2 = A_{11}\sin^2\theta + A_{33}\cos^2\theta = A_{33} + (A_{11}-A_{33})\sin^2\theta$$

$$(4.3) \qquad V_{SV} = \sqrt{A_{55}}$$

4.2 Ray and Normal Velocities

The angle and velocity that are most often considered in seismological applications are the ones describing the ray and not those of the wavefront normal associated with the ray. It is the latter of these two which has been considered until now, and except for spherical wavefronts, the two are in general not the same. As can be seen from equation (4.3) the normal velocity of the SV type wavefront is independent of any

angle, from which it follows that the SV type wavefront is forced to be spherical and hence the ray velocity and normal velocity are the same in magnitude and direction.

Using Euler's theorem on homogeneous functions, the nonlinear partial differential equation $G_m(p_i)=1$, (m=SV or P) can be solved by means of characteristics (Courant and Hilbert (4)). The equations of the characteristics corresponding to the partial differential equation $G_m(p_i)=1$ are given by

$$\frac{dx_i}{dt} = \frac{1}{2}\frac{\partial G_m(p_i)}{\partial p_i} \tag{4.4}$$

The quantities dx_i/dt are the components of the ray velocity.

With the simplification (4.1) these velocity components are

$$\begin{aligned} \frac{dx_1}{dt} &= p_1 A_{11} = \frac{A_{11} \sin\theta}{V_p} \\ \frac{dx_3}{dt} &= p_3 A_{33} = \frac{A_{33} \cos\theta}{V_p} \end{aligned} \tag{4.5}$$

for P type rays, and

$$\frac{dx_1}{dt} = p_1 A_{55} = \frac{A_{55} \sin \theta}{V_{sv}}$$

(4.6)

$$\frac{dx_3}{dt} = p_3 A_{55} = \frac{A_{55} \sin \theta}{V_{sv}}$$

for SV type rays. The angle θ is again the angle the wavefront normal makes with the vertical (x_3) axis.

The ratio of the ray velocity components yields the tangent of the angle, ϕ, the ray makes with the vertical axis.

(4.7) $\tan \phi = \frac{A_{11}}{A_{33}} \tan \theta$ for P type rays, and

(4.8) $\tan \phi = \tan \theta$ for SV type rays.

From (4.6) and (4.8) it is easily seen that the SV type ray velocity and angle are identical to the SV type normal velocity and angle. Using (4.5) and (4.7) the P type ray velocity $c = \left(\left(\frac{dx_1}{dt}\right)^2 + \left(\frac{dx_3}{dt}\right)^2\right)^{1/2}$ can be determined.

(4.9) $$\frac{1}{c^2} = \frac{\sin^2\phi}{A_{11}} + \frac{\cos^2\phi}{A_{33}} = \frac{V_p^2}{A_{11}^2 \sin^2\theta + A_{33}^2 \cos^2\theta}$$

V_p being the normal velocity of the P type wavefront.

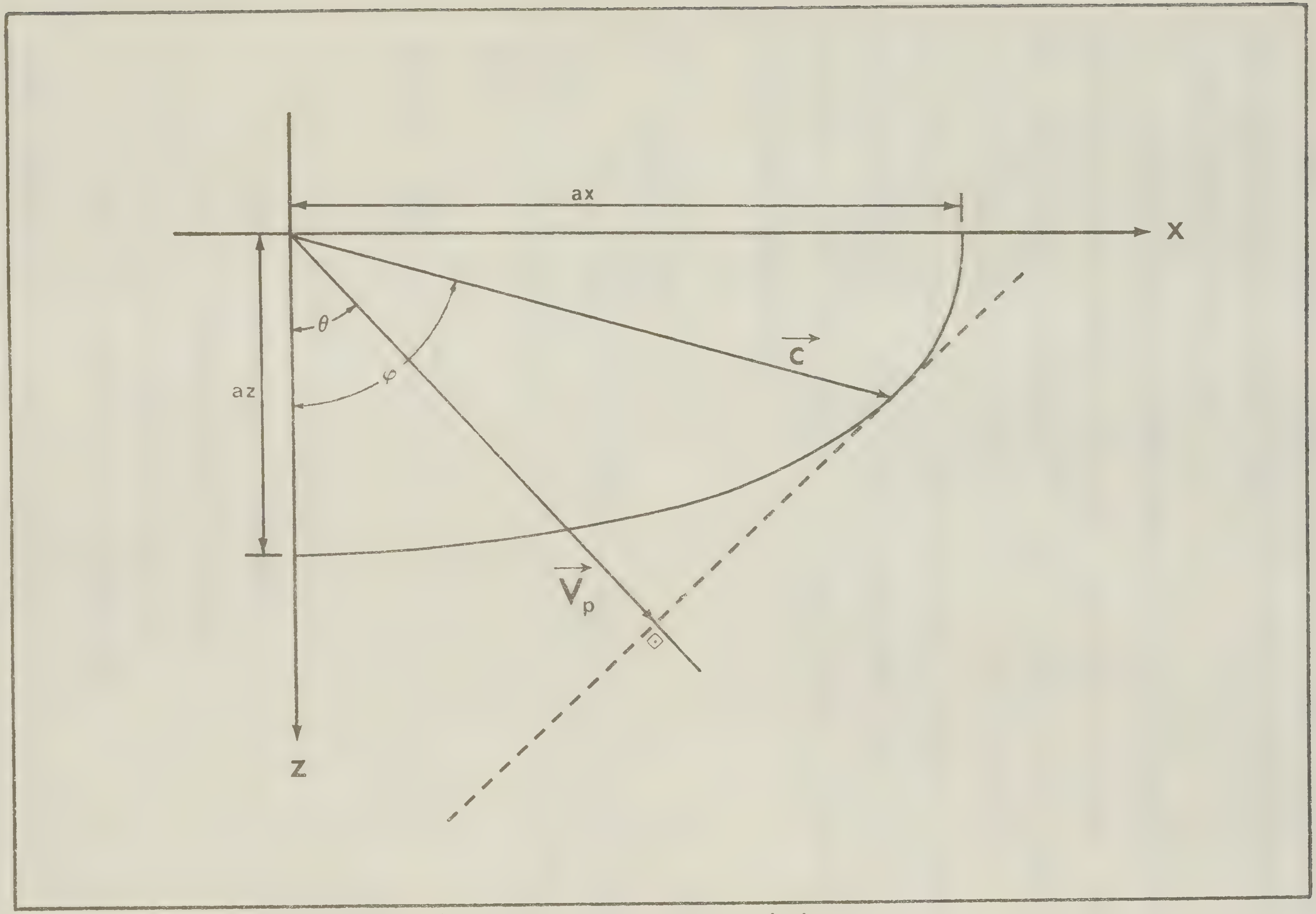


Fig. A4 Relation between normal and ray velocities.

The relation between wavefront normal angles and velocities and ray angles and velocities can be seen graphically in Fig. A4.

It should be noted that at $\theta = \phi = 0$ and $\theta = \phi = \pi/2$ the normal velocity and ray velocity are equal; i.e. $c^o = V_p^o = \sqrt{A_{33}} = az$ and $c^{\pi/2} = V_p^{\pi/2} = \sqrt{A_{11}} = ax$. It is assumed that these two velocities can be measured as can $\sqrt{A_{55}} = b$, the SV ray and wavefront velocity, along with the density. With these parameters known, the medium can, for the puposes of this thesis, be considered fully described.

4.3 Snell's Law

The simpler expressions obtained for normal velocities can be substituted into the statement of Snell's law

$$\frac{\sin \theta_o}{V_o} = \frac{\sin \theta_v}{V_o} \tag{4.10}$$

v=0 - incident wavefront

v=1 - reflected P wavefront

v=2 - transmitted P wavefront

v=3 - reflected SV wavefront

v=4 - transmitted SV wavefront

and if the angle of the incident wavefront is known, expressions can easily be obtained for the angle of the

reflected and transmitted wavefront normals.

As an example, consider a P type wavefront incident from medium 1 on an interface. The angle of the transmitted P type wavefront normal in medium 2 is given by

$$\sin \theta_2 = az2 \sin \theta_o / \{az1^2 + (ax1^2 - az1^2 - ax2^2 + az2^2) \times \sin^2\theta_o\}^{\frac{1}{2}} \quad . \tag{4.11}$$

In a similar manner all other wavefront normal angles can be expressed in terms of the sine of the angle of the incident and/or reflected P wavefront normal.

4.4 Numerical Results

A program was written using Fortran IV which calculates reflection and transmission coefficients, including the free interface case and surface conversion coefficients, using the formulas developed earlier in the paper for varying angles of ray incidence.

A Calcomp plotter was used to display the results which consist of two graphs for each coefficients; one of modulus of the coefficient vs. ray angle of incidence and the other with phase of the coefficient vs. ray angle of incidence, since in general the coefficients are complex. (The phase in the following plots may in some cases vary by π, since in writing the program, the z-axis was shown to point downwards.)

Since several situations are to be investigated space dictates that not all coefficients can be presented. The four coefficients that will be compared for each of these situations are P1P1, P1P2, S1S1, S1P2.

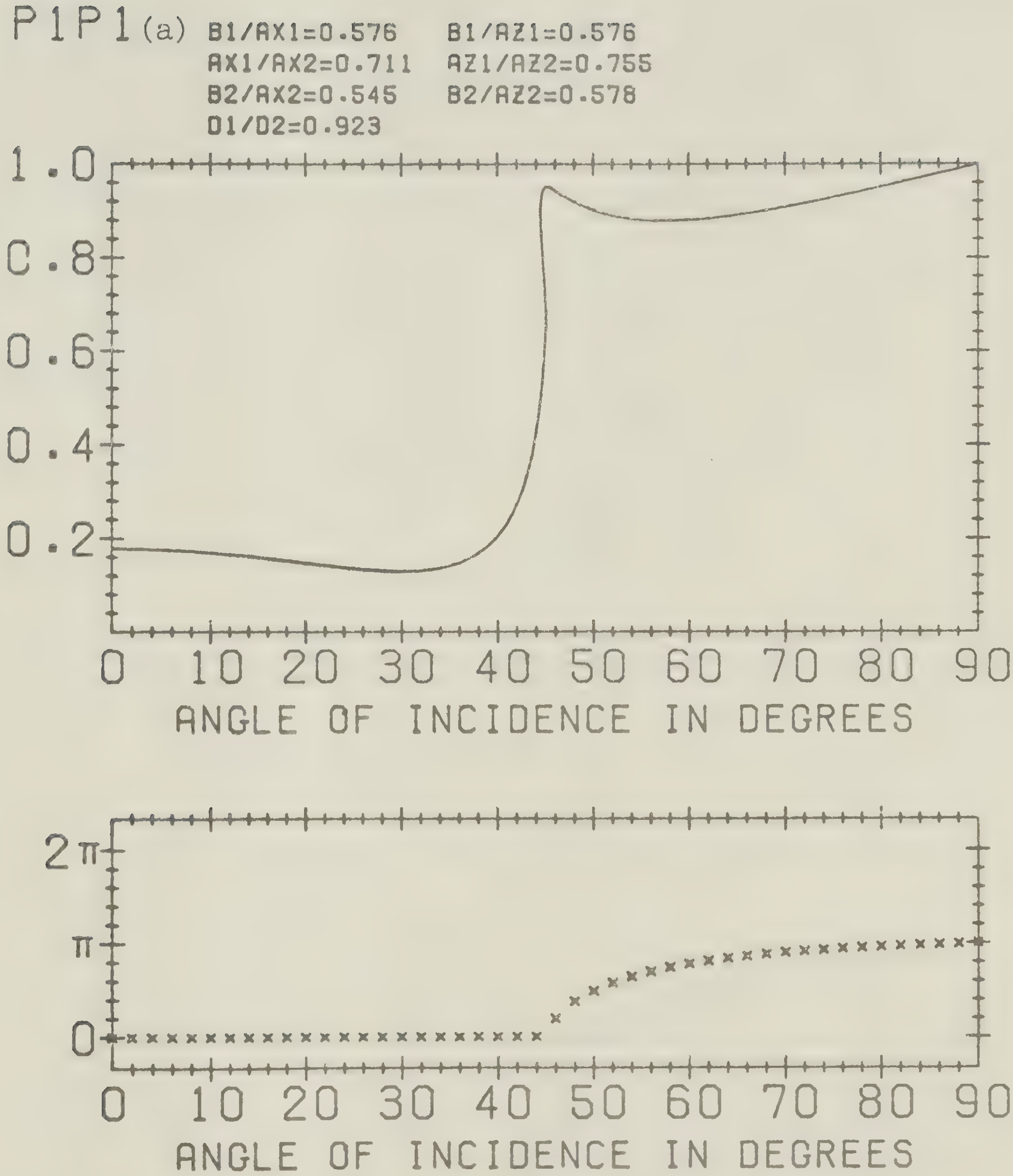
The situations that will be looked at are (a) isotropic medium over a transversely isotropic medium (b) transversely isotropic medium over an isotropic medium (c) transversely isotropic medium over a transversely isotropic medium (d) liquid medium over a transversely isotropic medium. The parameters of the media are tabulated below. (The velocities given should not be taken as absolute as it is only their ratios that are of significance.)

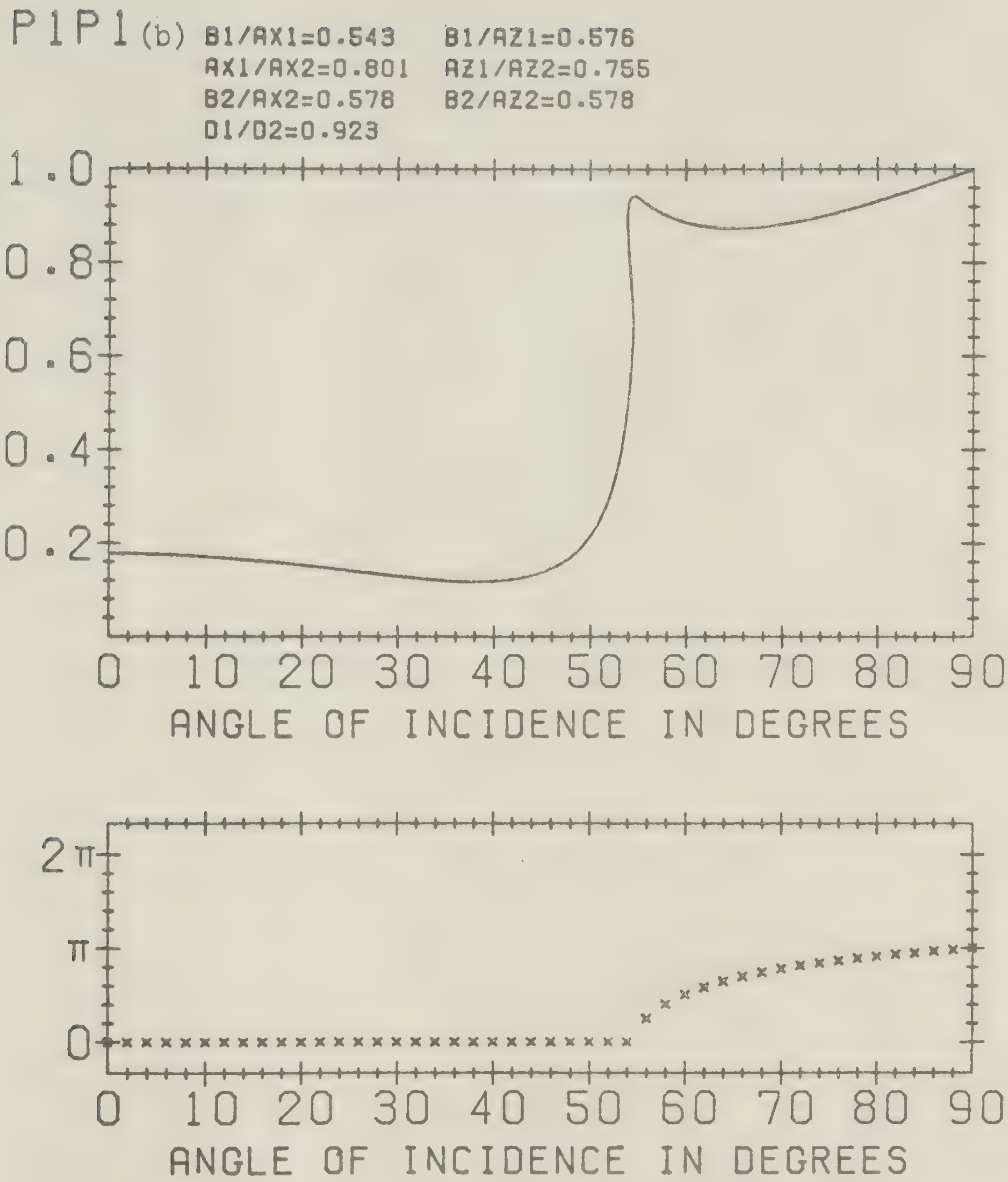
A5. Description of the Media

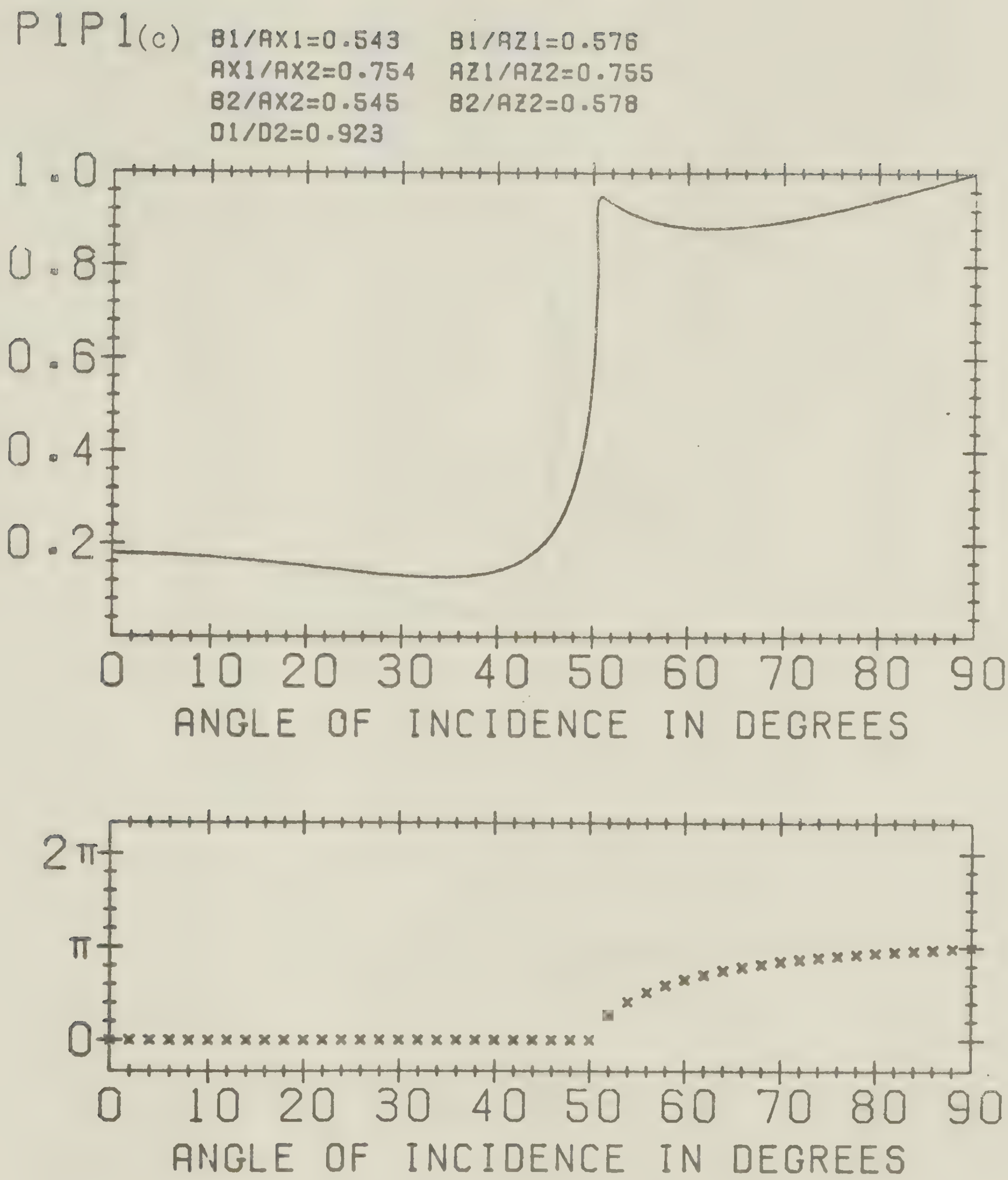
(a)	AX1 = 2.31 km/sec	AX2 = 3.25 km/sec
	AZ2 = 2.31 km/sec	AZ2 = 3.06 km/sec
	B1 = 1.33 km/sec	B2 = 1.77 km/sec
	D1 = 2.04 gm/cc	D2 = 2.21 gm/cc
(b)	AX1 = 2.45 km/sec	AX2 = 3.06 km/sec
	AZ1 = 2.31 km/sec	AZ2 = 3.06 km/sec
	B1 = 1.33 km/sec	B2 = 1.77 km/sec
	D1 = 2.04 gm/cc	D2 = 2.21 gm/cc
(c)	AX1 = 2.45 km/sec	AX2 = 3.25 km/sec
	AZ1 = 2.31 km/sec	AZ2 = 3.06 km/sec
	B1 = 1.33 km/sec	B1 = 1.77 km/sec
	D1 = 2.04 km/sec	D1 = 2.21 gm/sec
(d)	AX1 = 2.31 km/sec	AX2 = 3.25 km/sec
	AZ1 = 2.31 km/sec	AZ2 = 3.06 km/sec
	B1 = 0.00 km/sec	B2 = 1.77 km/sec
	D1 = 2.04 gm/cc	D2 = 2.21 km/sec

Figure A6

P1P1 reflection coefficients


P1P1 (a)
B1/AX1=0.576 B1/AZ1=0.576
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


P1P1 (b)
B1/AX1=0.543
B1/AZ1=0.576
AX1/AX2=0.801
AZ1/AZ2=0.755
B2/AX2=0.578
B2/AZ2=0.578
D1/D2=0.923
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


P1P1(c)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.754 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

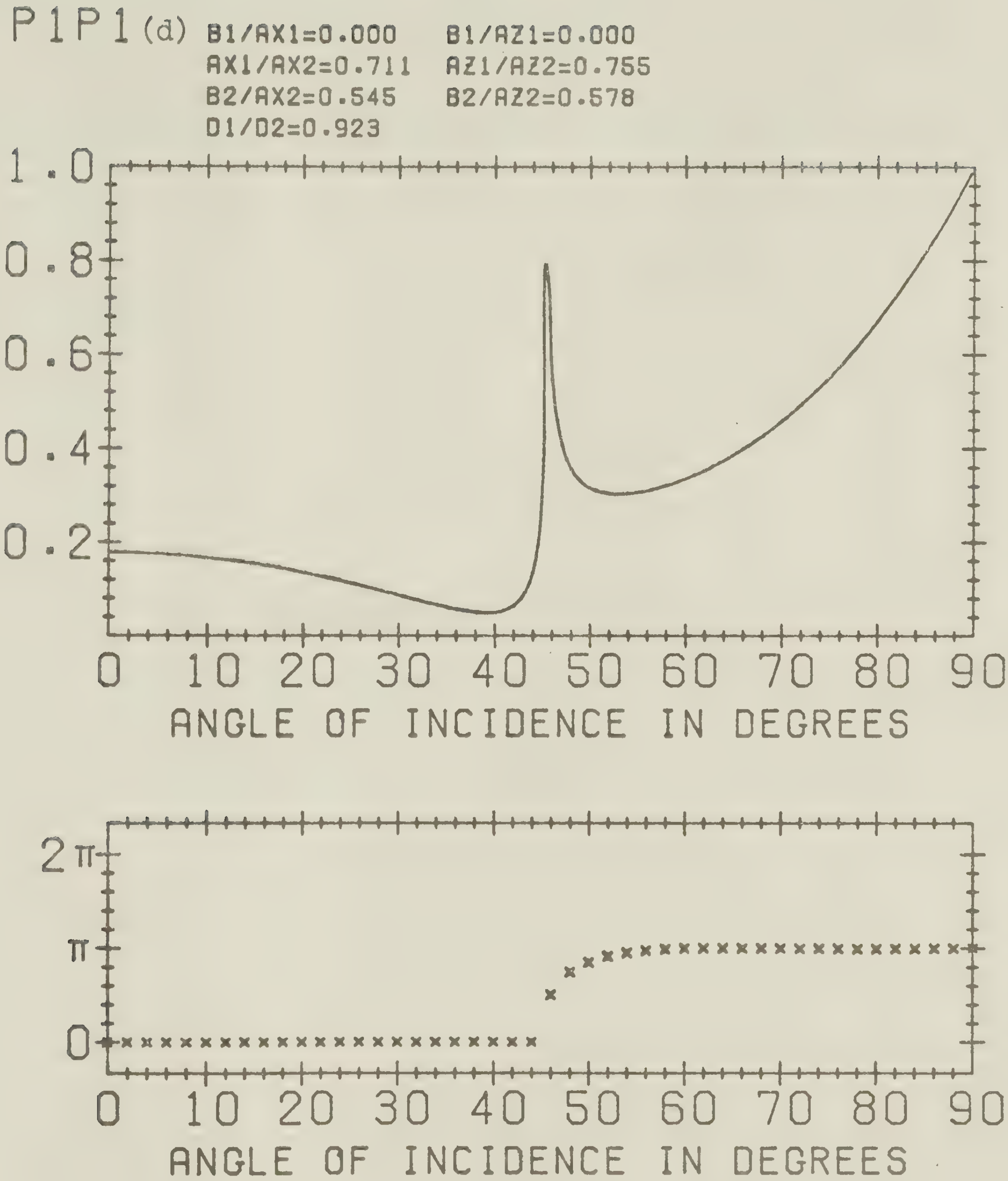
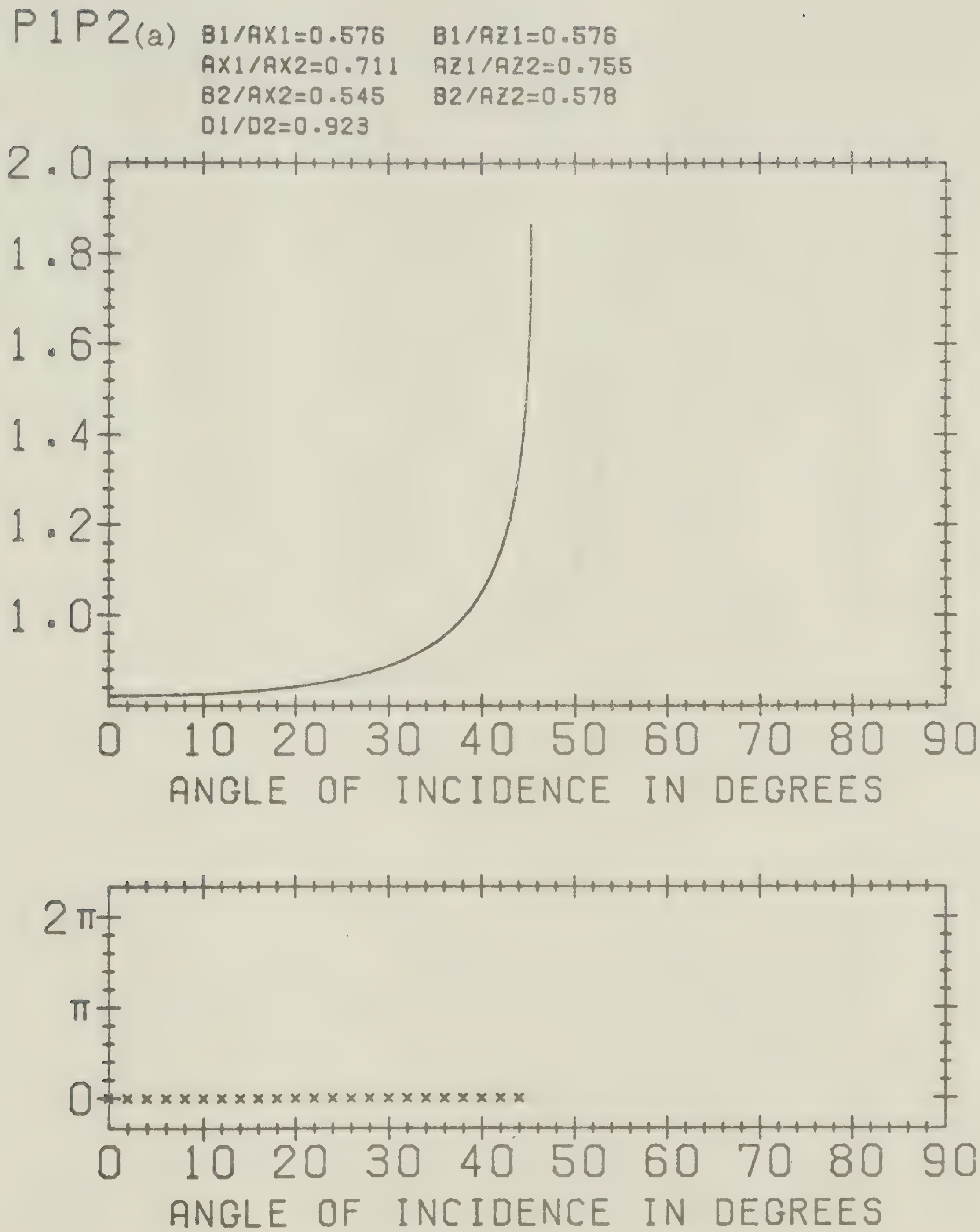
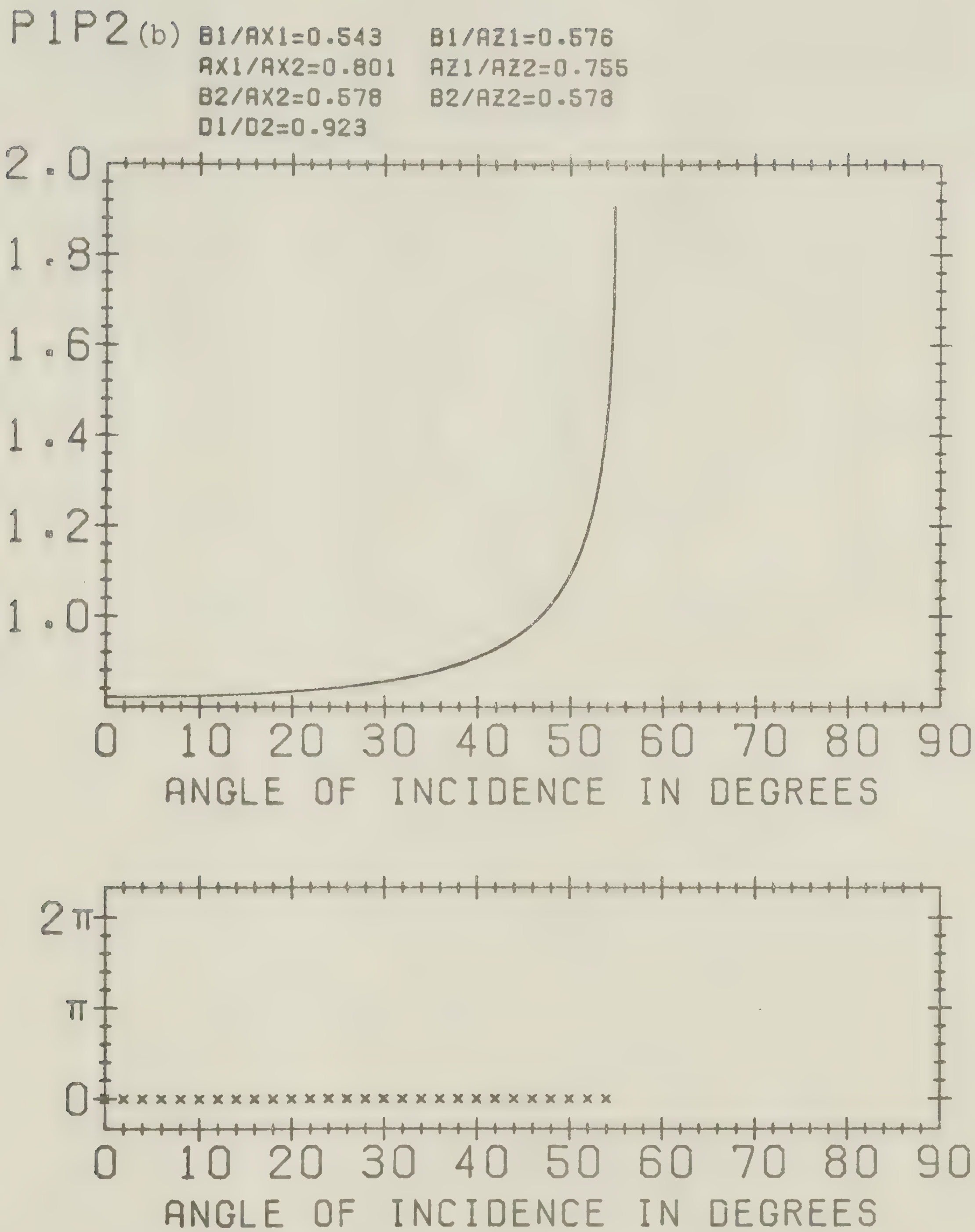
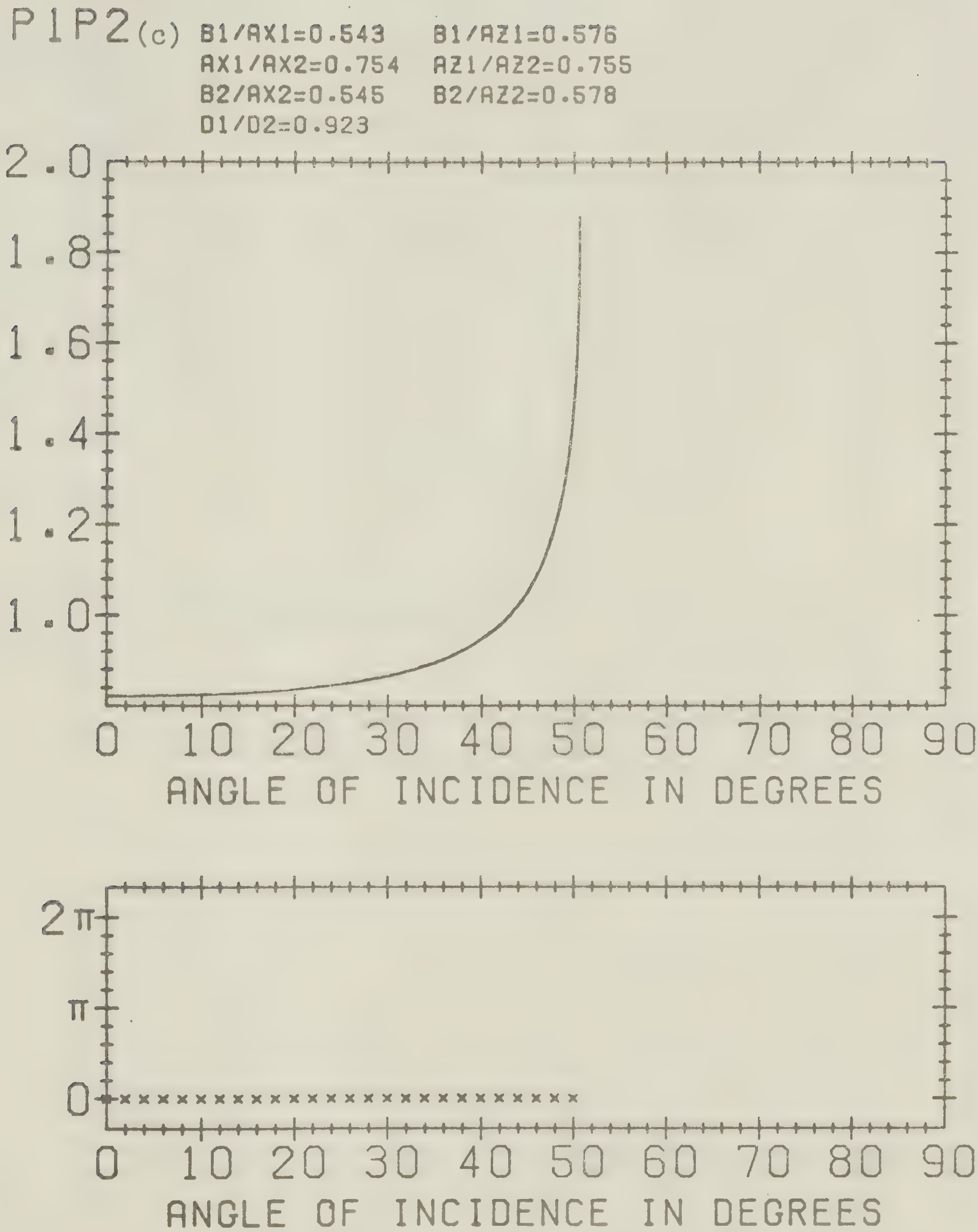

P1P1 (d)
B1/AX1=0.000 B1/AZ1=0.000
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

Figure A7

P1P2 transmission coefficients


P1P2(a)
B1/AX1=0.576 B1/AZ1=0.576
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
2.0
1.8
1.6
1.4
1.2
1.0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


P1P2 (b)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.801 AZ1/AZ2=0.755
B2/AX2=0.578 B2/AZ2=0.578
D1/D2=0.923
2.0
1.8
1.6
1.4
1.2
1.0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


P1P2(c)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.754 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
2.0
1.8
1.6
1.4
1.2
1.0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

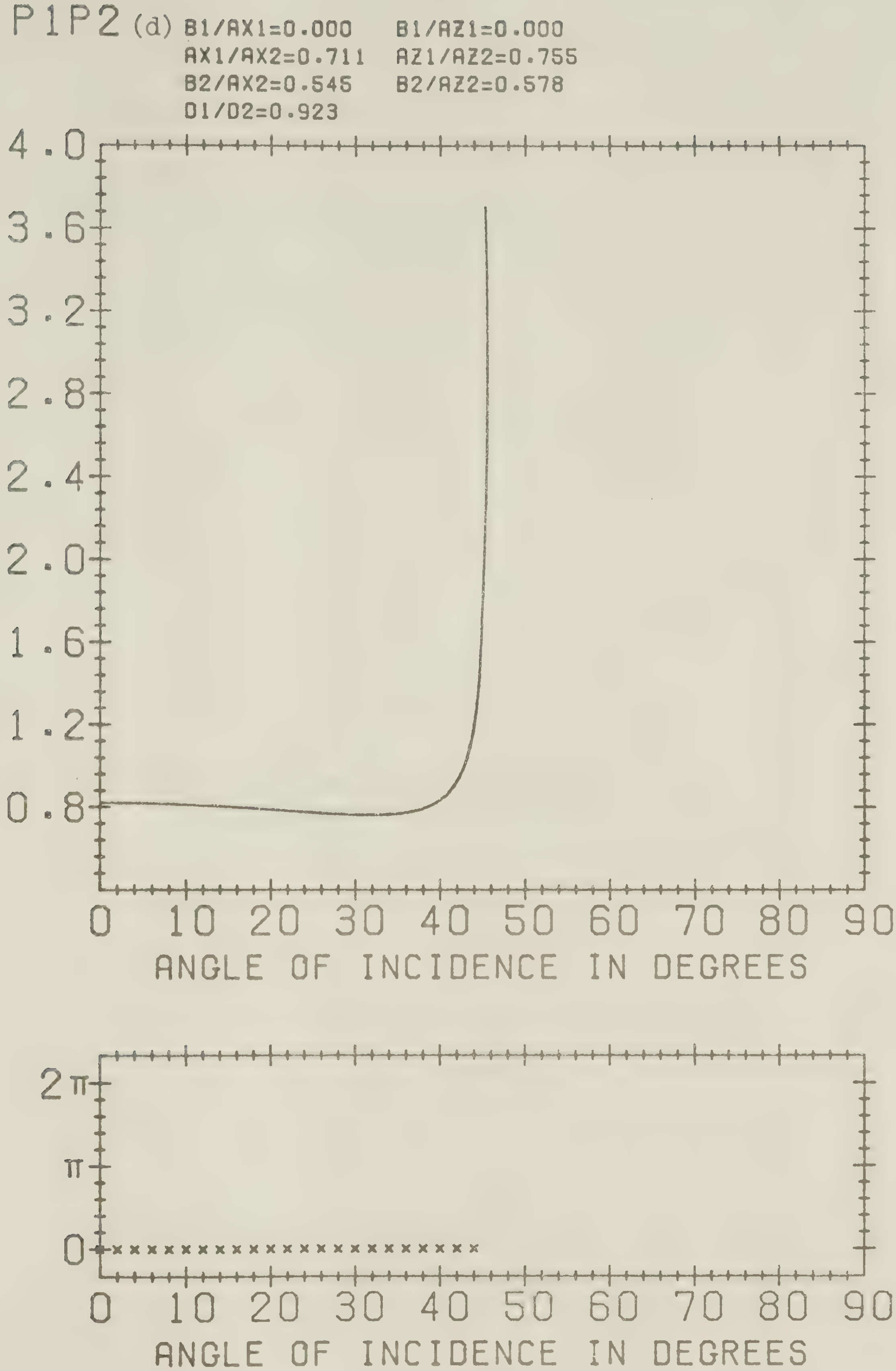
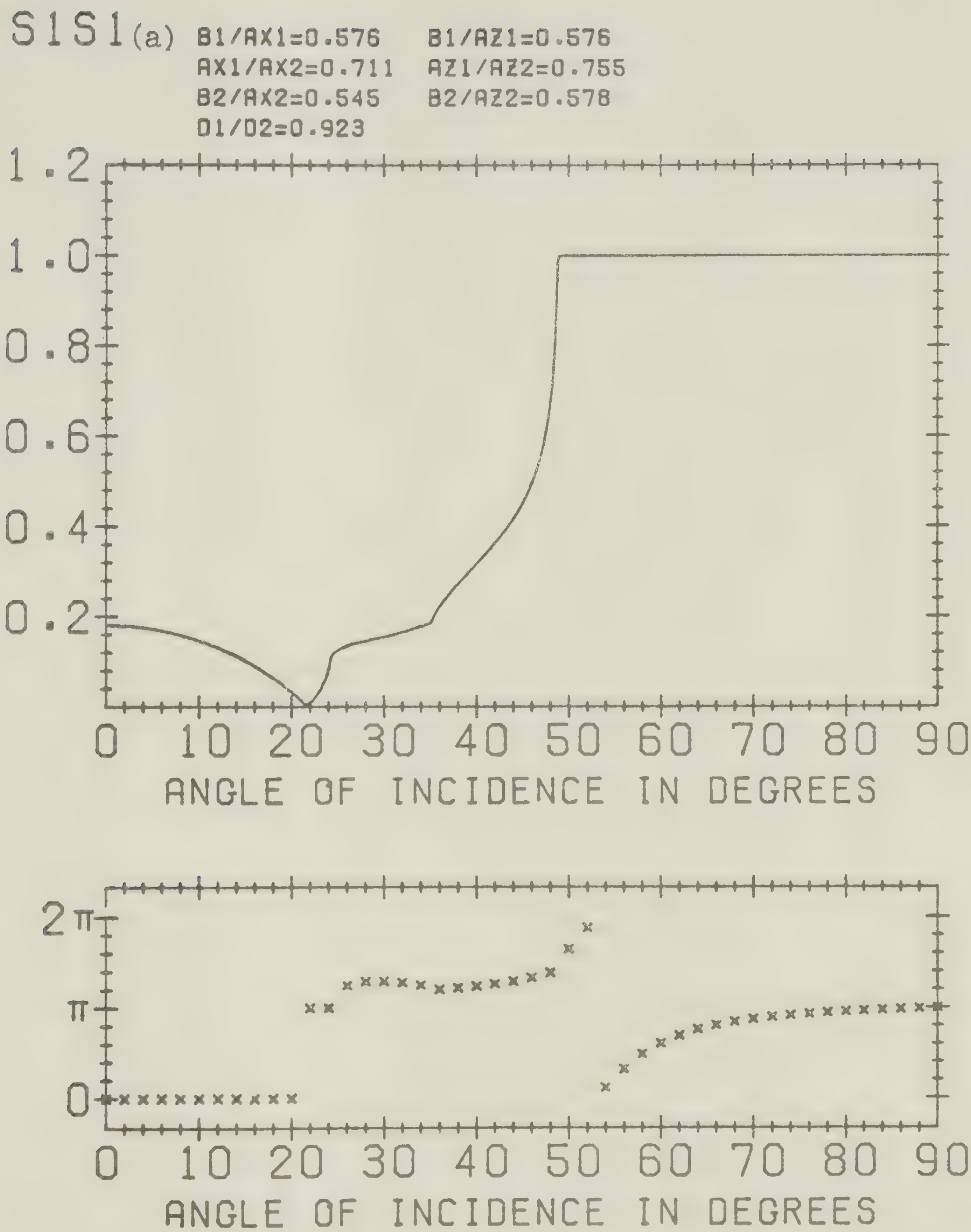
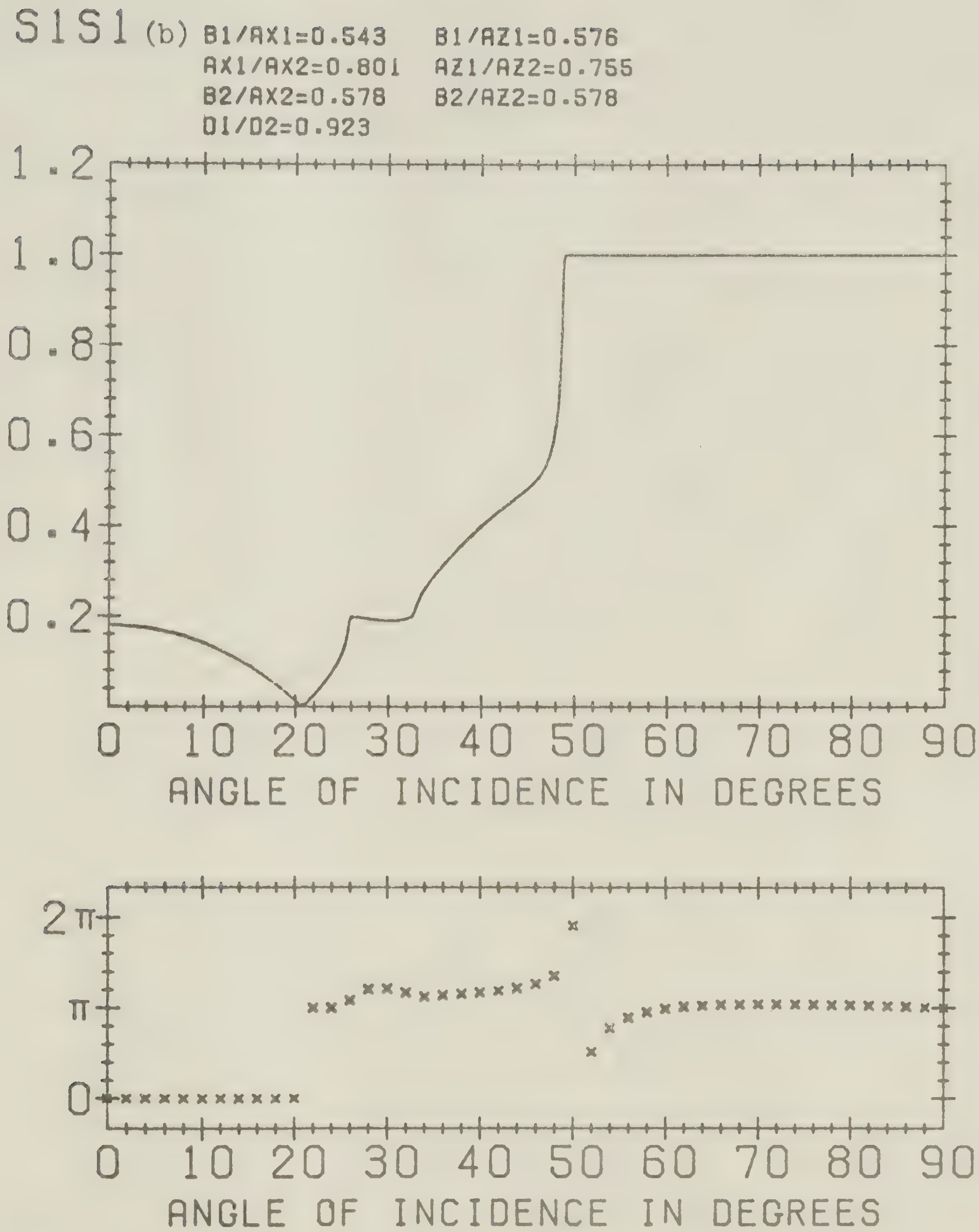

P1P2 (d)
B1/AX1=0.000 B1/AZ1=0.000
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
4.0
3.6
3.2
2.8
2.4
2.0
1.6
1.2
0.8
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

Figure A8

S1S1 reflection coefficients


S1S1(a)
B1/AX1=0.576 B1/AZ1=0.576
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
1.2
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


S1S1 (b)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.801 AZ1/AZ2=0.755
B2/AX2=0.578 B2/AZ2=0.578
D1/D2=0.923
1.2
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

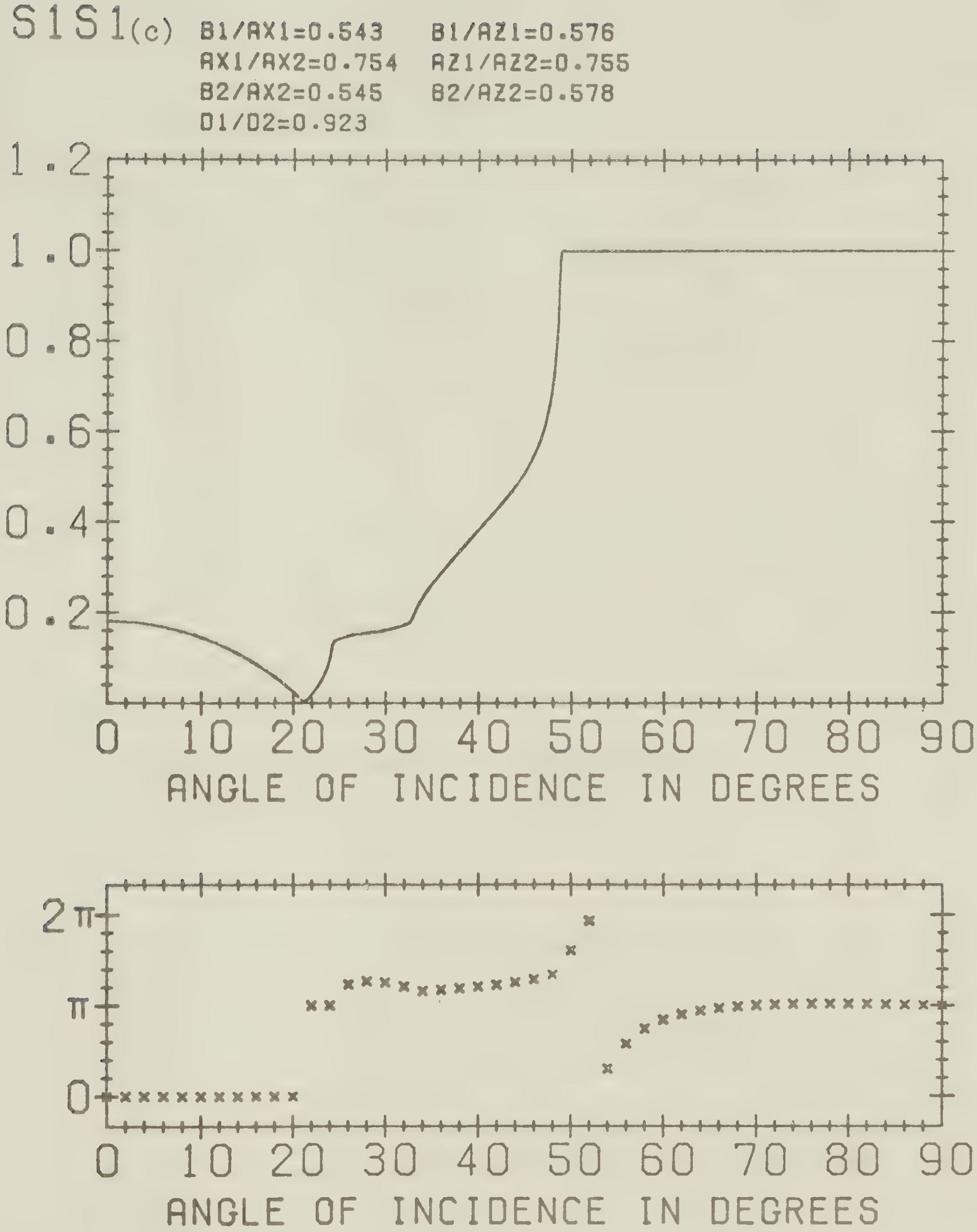
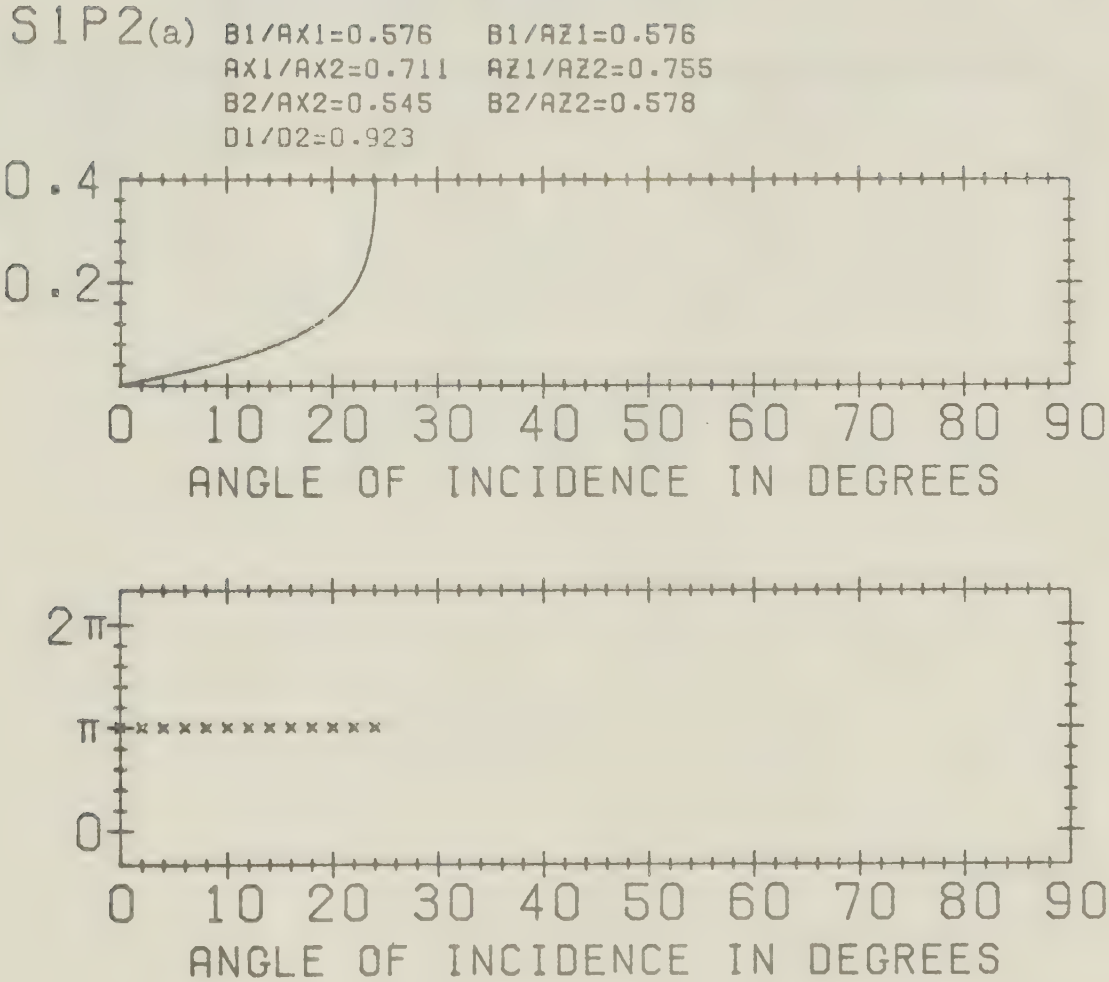
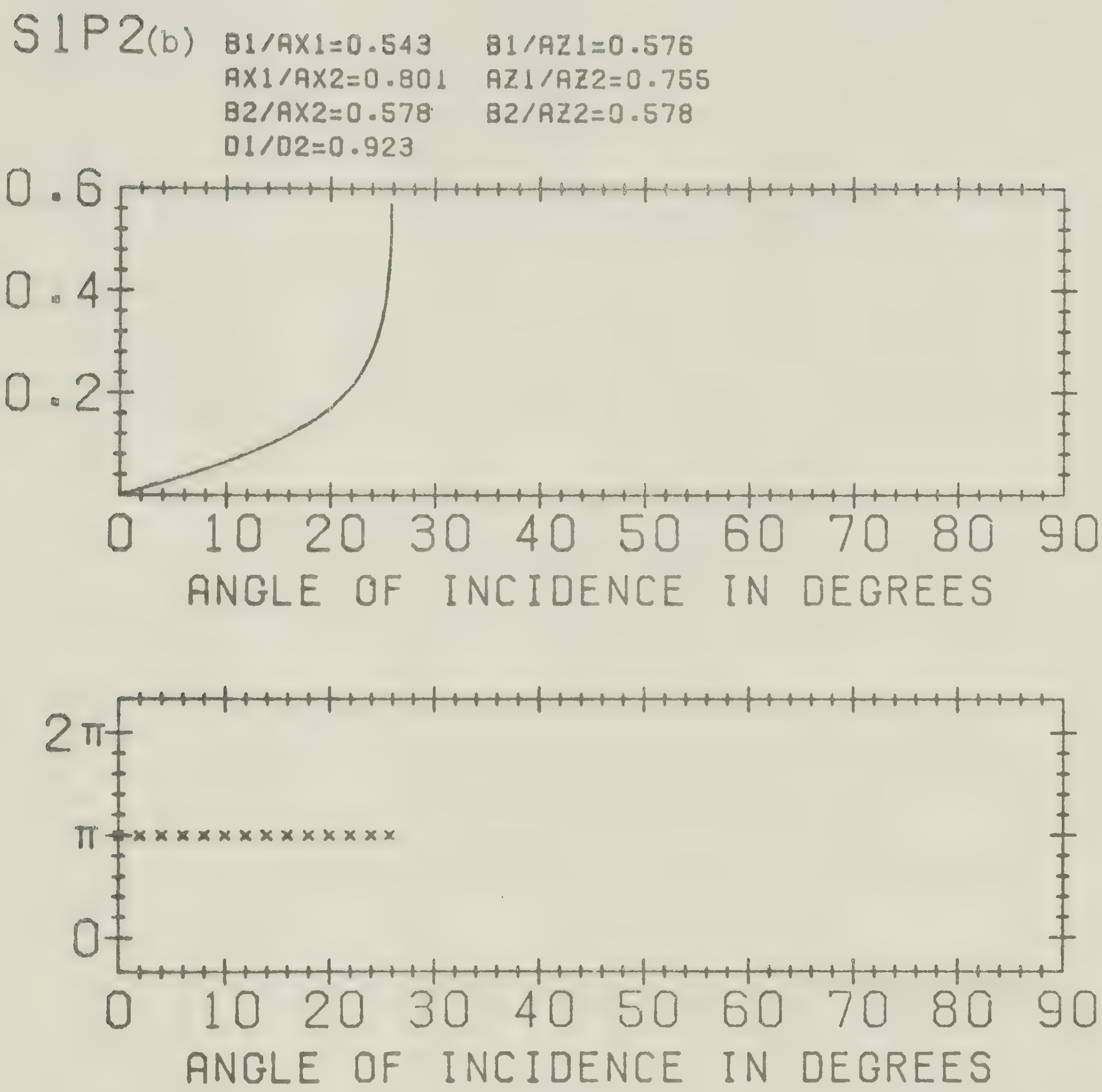
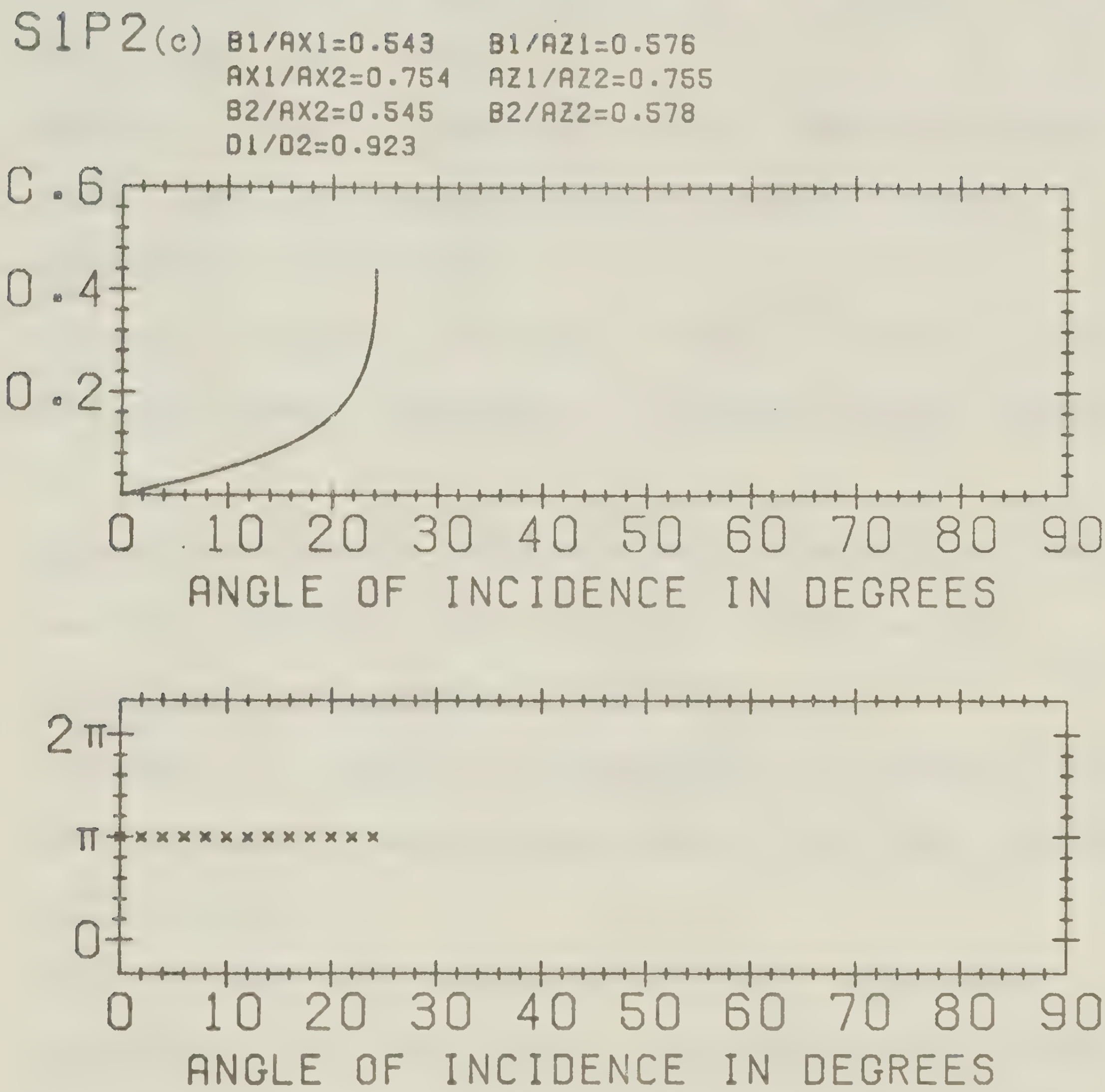

S1S1(c)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.754 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
1.2
1.0
0.8
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

Figure A9

S1P2 transmission coefficients


S1P2(a)
B1/AX1=0.576 B1/AZ1=0.576
AX1/AX2=0.711 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES


S1P2(b)
B1/AX1=0.543
B1/AZ1=0.576
AX1/AX2=0.801
AZ1/AZ2=0.755
B2/AX2=0.578
B2/AZ2=0.578
D1/D2=0.923
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0
10
20
30
40
50
60
70
80
90
ANGLE OF INCIDENCE IN DEGREES


S1P2(c)
B1/AX1=0.543 B1/AZ1=0.576
AX1/AX2=0.754 AZ1/AZ2=0.755
B2/AX2=0.545 B2/AZ2=0.578
D1/D2=0.923
0.6
0.4
0.2
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES
2π
π
0
0 10 20 30 40 50 60 70 80 90
ANGLE OF INCIDENCE IN DEGREES

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APPENDIX

Formulae Pertaining to the Reflection and Transmission Coefficients

The $E_{ij} = 1, 18$ used in the coefficients are given below.

$$
\text{Ap. 1} \quad
\begin{cases}
E_1 = T_1 T_2 x^2 \\
E_2 = T_3 T_4 PQ \\
E_3 = T_5 T_6 PR \\
E_4 = T_7 T_8 x^2 PQRS \\
E_5 = T_9 T_{10} QS \\
E_6 = T_{11} T_{12} RS \\
E_7 = 2T_5 T_{11} PR \\
E_8 = 2T_2 T_{10} PQ \\
E_9 = -2x T_7 T_{11} PRS \\
E_{10} = -2x T_1 T_3 P \\
E_{11} = -2x T_7 T_{10} PQS \\
E_{12} = 2x T_1 T_5 P \\
E_{13} = -2x T_8 T_{12} QRS \\
E_{14} = -2x T_2 T_4 Q \\
E_{15} = 2x T_5 T_8 PQR \\
E_{16} = -2x T_2 T_{10} Q \\
E_{17} = 2T_{10} T_{12} QS \\
E_{18} = 2T_4 T_5 PQ
\end{cases}
$$

where

$$
\left\{
\begin{aligned}
T_1 &= \varepsilon_2 - \frac{\varepsilon_1 \ell_2}{\eta \ell_1} \\
T_2 &= \beta_2 \omega_2 k_1 \frac{\ell_3}{m_1} - \beta_1 \frac{\omega_1 k_2 \ell_4}{n \quad m_1} \\
T_3 &= \beta_2 \omega_2 + \beta_1 \frac{k_2 x^2 \ell_4}{n \quad m_1} \\
T_4 &= \varepsilon_2 \frac{m_3}{\ell_1} + \frac{\delta_1 x^2 \ell_2}{n \; \ell_1} \\
T_5 &= \beta_1 \left(\omega_1 + k_1 x^2 \frac{\ell_3}{m_1}\right) \\
T_6 &= \varepsilon_2 \frac{m_4}{\ell_1} + \frac{\delta_2 x^2}{n} \frac{\ell_2}{\ell_1} \\
T_7 &= \beta_2 \ell - \beta_1 \frac{m_2}{m_1} \\
T_8 &= \delta_2 \frac{m_3}{\ell_1} - \delta_1 \frac{m_4}{\ell_1} \\
T_9 &= \beta_2 \left(\omega_2 \frac{m_2}{m_1} + \frac{k_2 x^2 \ell \ell_4}{n \; m_1}\right) \\
T_{10} &= \varepsilon_1 \frac{m_3}{\ell_1} + \delta_1 x^2
\end{aligned}
\right.
\tag{Ap. 2}
$$

$$\begin{cases} T_{11} = \varepsilon_1 \dfrac{m_4}{\ell_1} + \delta_2 x^2 \\[2ex] T_{12} = \beta_2 \dfrac{k_1 x^2 \ell \ell_3}{m_1} + \dfrac{\beta_1 \omega_1 m_2}{m_1} \end{cases}$$

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